Matter from Space

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"People slowly accustomed themselves to the idea that the physical states of space itself were the final physical reality."

—PROFESSOR ALBERT EINSTEIN

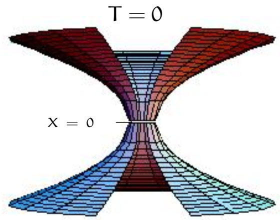
The New Yorker

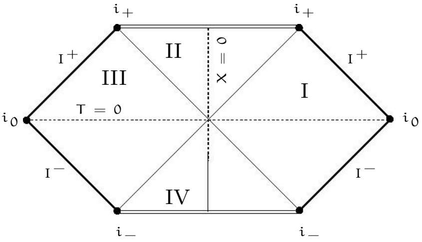
William Kingdon Clifford 1870

"I wish here to indicate a manner in which these speculations may be applied to the investigation of physical phenomena. I hold in fact:

1. That small portions of space *are* in fact of a nature analogous to little hills on a surface which is on the average flat; namely, that the ordinary laws of geometry are not valid in them.
2. That this property of being curved or distorted is continually being passed from one portion of space to another after the manner of a wave.
3. That this variation of the curvature of space is what really happens in that phenomenon which we call the *motion of matter*, whether ponderable or etherial.
4. That in the physical world nothing else takes place but this variation, subject (possibly) to the law of continuity."

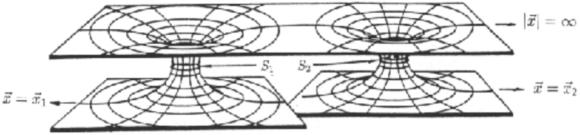
The $\mathbb{RP}^3$ geon

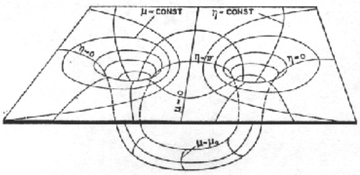


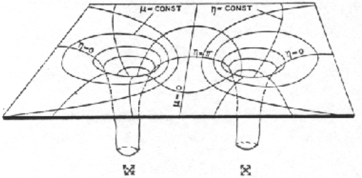


$$(T, X, \theta, \varphi) \mapsto (T, -X, \pi - \theta, \varphi + \pi)$$

Topologies for two BHs





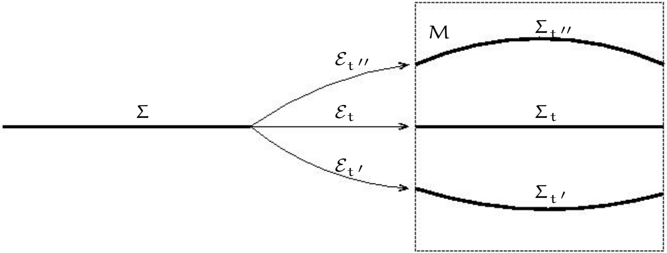


What is a "Body" ?

"In conclusion, the geons make only this contribution to science: it completes the scheme of classical physics by providing for the first time an acceptable classical theory of the concept of body."

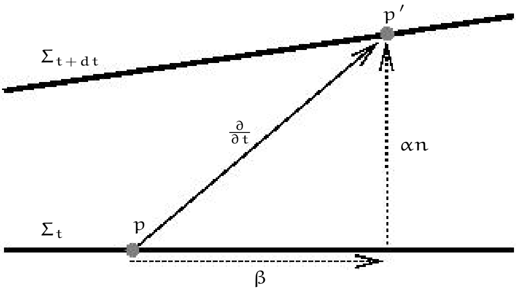
J.A. Wheeler in "Geons" (Phys. Rev. 97, 1955)

Spacetime as space's history



Spacetime, M, is foliated by a one-parameter family of embeddings $\mathcal{E}_t$ of the 3-manifold Σ into M. Σ_t is the image in M of Σ under $\mathcal{E}_t$.

A four-function worth of arbitrariness



For $q \in \Sigma$ the image points $p = \mathcal{E}_t(q)$ and $p' = \mathcal{E}_{t+dt}(q)$ are connected by the vector $\partial/\partial t|_p$ whose components tangential and normal to Σ_t are β (three functions) and αn (one function) respectively.

Kinematics of hypersurface deformations

- In local coordinates y^μ of M and x^m of Σ the generators of normal and tangential deformations of the embedded hypersurface are

$$\begin{aligned} N_\alpha &= \int_\Sigma d^3x\ \alpha(x)\, n^\mu[y(x)]\, \frac{\delta}{\delta y^\mu(x)} \\ T_\beta &= \int_\Sigma d^3x\ \beta^m(x)\, \partial_m y^\mu(x)\, \frac{\delta}{\delta y^\mu(x)} \end{aligned}$$

- This is merely the foliation-dependent decomposition of the tangent vector $X(V)$ at $y \in \mathrm{Emb}(\Sigma, M)$, induced by the spacetime vector field $V = \alpha n + \beta^a \partial_a$:

$$X(V) = \int_\Sigma d^3x\ V^\mu(y(x))\, \frac{\delta}{\delta y^\mu(x)}$$

- The vector fields $X(V)$ on $\text{Emb}(\Sigma, M)$ obey

$$[X(V), X(W)] = X\big([V, W]\big) \,,$$

  i.e. $V \mapsto X(V)$ is a Lie homomorphism from the tangent-vector fields on M to the tangent-vector fields on $\text{Emb}(\Sigma, M)$.
- In terms of the normal-tangential decomposition:

$$\begin{aligned}
[T_\beta \,,\, T_{\beta'}] &= -T_{[\beta,\beta']} \,, \\
[T_\beta \,,\, N_\alpha] &= -N_{\beta(\alpha)} \,, \\
[N_\alpha \,,\, N_{\alpha'}] &= -\epsilon\, T_{\alpha\, \mathrm{grad}_h(\alpha') - \alpha' \mathrm{grad}_h(\alpha)} \,,
\end{aligned}$$

- Here $\epsilon = 1$ for Lorentzian and $= -1$ for Euclidean spacetimes, just to keep track of signature dependence.

Hamiltonian Geometrodynamics

- The idea is to let Hamiltonian represent hypersurface deformations (Poisson action)
- **Theorem (Teitelboim, Kuchař 1973-4):** The most general local realisation on the cotangent bundle over Riem(Σ), coordinatised by (h, π), is

$$N_\alpha \mapsto H_\alpha[h,\pi] := \int_\Sigma \alpha(x)\,\mathcal{H}[h,\pi](x)$$

$$T_\beta \mapsto D_\beta[h,\pi] := \int_\Sigma \beta^a(x)\,h_{ab}(x)\,\mathcal{D}^b[h,\pi](x)$$

where

$$\mathcal{H}[h,\pi] := \epsilon(2\kappa)G_{ab\,cd}\pi^{ab}\pi^{cd} - (2\kappa)^{-1}\sqrt{h}\,(R-2\Lambda)$$

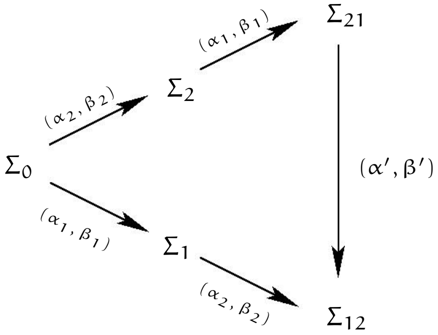
$$\mathcal{D}^b[h,\pi] := -2\nabla_a\pi^{ab}$$

with (5+1) Lorentzian Wheeler - De Witt metric on momenta:

$$G_{ab\,cd} = \left(h_{ac}h_{bd} + h_{ad}h_{bc} - \lambda\, h_{ab}h_{cd}\right)/2\sqrt{h}$$

and $\lambda = 1$ (hence required by 4-d "path independence").

Commutators



Successive hypersurface deformations parametrised by (α_1, β_1) and $N_2 = (\alpha_2, \beta_2)$ do not commute; rather

$$[X(\alpha_1, \beta_1), X(\alpha_2, \beta_2)] = X(\alpha', \beta'),$$

where

$$\begin{aligned} \alpha' &= \beta_1(\alpha_2) - \beta_2(\alpha_1), \\ \beta' &= [\beta_1, \beta_2] + \alpha_1 \operatorname{grad}_h(\alpha_2) - \alpha_2 \operatorname{grad}_h(\alpha_1). \end{aligned}$$

Need for constraints

- Since α' depends on h, we get the following condition for the Hamiltonians to act (via Poisson Bracket) as derivations on phase-space functions:

$$\begin{aligned}
&\{\{F, H(\alpha_1, \beta_1)\}, H(\alpha_2, \beta_2)\} - \{\{F, H(\alpha_2, \beta_2)\}, H(\alpha_1, \beta_1)\} \\
&= \{F, \{H(\alpha_1, \beta_1), H(\alpha_2, \beta_2)\}\} = \{F, H(\alpha', \beta')\} \\
&= \{F, H\}(\alpha', \beta') + H\big(\{F, \alpha'\}, \{F, \beta'\}\big) \\
&\stackrel{!}{=} \{F, H\}(\alpha', \beta')
\end{aligned}$$

- The last equality must hold for all F and all (α_1, β_1) (α_2, β_2). This implies the constraints:

$$\mathcal{H}[h, \pi](x) = 0 \qquad \mathcal{D}^a[h, \pi](x) = 0$$

- Constraints correspond to $\perp\perp$ and $\perp \parallel$ components of Einstein's equation. A spacetime in which constraints are satisfied for *each* Σ must obey Einstein's equation.
- The constraints do not cause topological obstructions to Cauchy surface. Only special requirements do, like e.g. time-symmetry.

Mass without mass

- The mass-energy of an asymptotically flat end is

$$m \propto \lim_{R \to \infty} \left\{ \int_{S^2_R \subset \Sigma} d\sigma(\partial_a h_{ab} - \partial_b h_{aa}) n^b \right\}$$

- This is ≥ 0 and $= 0$ for Minkowski slices only.
- Gannon's theorem implies causal geodesic incompleteness if $\pi_1(\Sigma) \neq 1$ (replacing $\exists$ trapped surfaces in the hypotheses).
- Stationary regular vacuum solutions (gravitational solitons) do not exist (Einstein & Pauli, Lichnerowicz).

  Proof: Positive-mass theorem and ADM = Komar for stationary space-times:

$$m \propto \int_{S^2_\infty} \star dK = \int_\Sigma \underbrace{d \star dK}_{\propto i_K \mathrm{Ric}}$$

Momenta without momenta

- The linear and angular momenta of an asymptotically flat end is

$$p^a \propto \int_{S^2_\infty} d\sigma\, \pi^{ab} n_b \,, \qquad J^a \propto \int_{S^2_\infty} d\sigma\, \varepsilon_{abc}\, x^b \pi^{cd} n_d$$

- Axisymmetric vacuum configurations with $J \neq 0$ and one end do not exist, even for non-orientable Σ:

$$J_K = \int_{S^2_\infty} \star dK = \int_\Sigma \underbrace{d \star dK}_{\propto i_K \mathrm{Ric}} = 0$$

- But for Killing fields K up to sign they do (Friedman & Mayer 1981).

Charge without charge

- Electrovac solutions with non-zero overall electric charge

$$Q_e = \int_{S^2_\infty} \star F$$

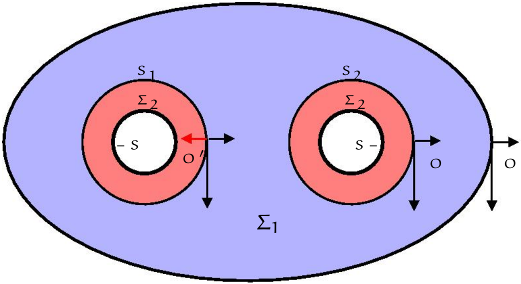
only exist if $S^2_\infty \neq \partial\Sigma$, i.e. if $[S^2_\infty] \in H^2(\Sigma)$ is non-trivial, like e.g. in Reissner-Nordström.

- If Σ has only one end and is *non-orientable*, Stokes' theorem obstructs existence of electric but not of magnetic charge (Sorkin 1977):

$$Q_m = \int_{S^2_\infty} F$$

- This is because for non-orientable Σ, Stokes' theorem holds for twisted (densitised) but not for ordinary forms.

Non orientable Wormehole



- Stokes' theorem applied to $\vec{\nabla} \cdot \vec{B} = 0$ in Σ_1:

$$\Phi(\vec{B}, \partial\Sigma_1, O) + \Phi(\vec{B}, S_1, O) + \Phi(\vec{B}, S_2, O) = 0$$

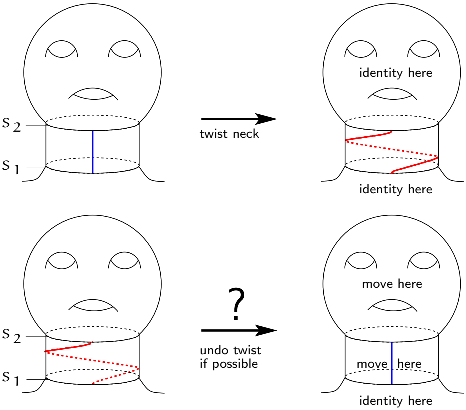
- Stokes' theorem applied to $\vec{\nabla} \cdot \vec{B} = 0$ in Σ_2:

$$\Phi(\vec{B}, S_1, O') + \Phi(\vec{B}, S_2, O) = 0$$

- Hence

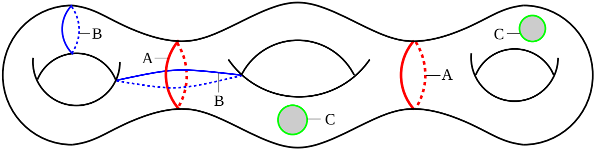
$$\Phi(\vec{B}, \partial\Sigma_1, O) = -2\,\Phi(\vec{B}, S_1, O) \neq 0$$

Spin without spin



- There exist many 3-manifolds for which a full (i.e. 2π) relative rotation is not in the id-component.
- In this case the asymptotic symmetry group at spacelike infinity contains $SU(2)$ rather than $SO(3)$.
- Mechanism 'fermions-from-bosons' in gravity? (Friedman & Sorkin 1982).
- M spinorial $\Leftrightarrow$ contains prime $\neq S^1 \times S^2$ and $\neq L(p, q)$.

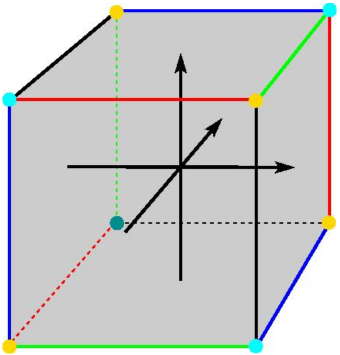
Connected sums



- Decompose along *splitting* and *essential* 2-spheres until only *prime-manifolds* remain. Prime factors are unique up to permutation.

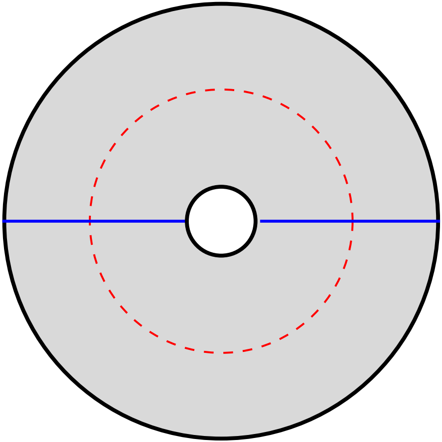
- Except for $S^1 \times S^2$, a prime manifold has trivial π_2. The converse is true given PC. Given TGC, all finite-π_1 primes are spherical space-forms S^3/G, $G \subset SO(4)$. Infinite-π_1 primes are $S^1 \times S^2$, the flat ones $\mathbb{R}^3/G$, $G \subset E_3$, and the huge family of locally hyperbolic ones.

Example: The space form S^3/D_8^*

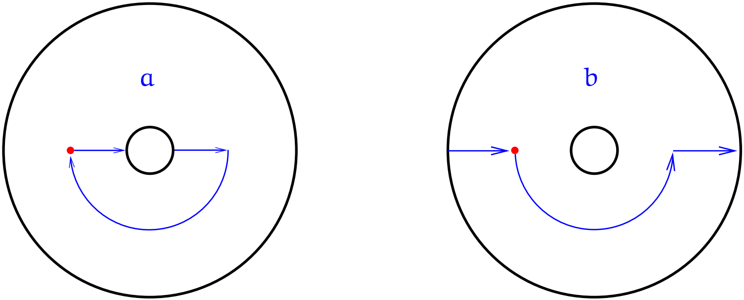


- $\Sigma = S^3/D_8^*$ is spinorial
- $D_8^* = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$
- $\mathrm{MCG}_\infty(\Sigma) \cong \mathrm{Aut}(D_8^*) \cong O$
- $\mathrm{MCG}_F(\Sigma) \cong \mathrm{Aut}_{\mathbb{Z}_2}(D_8^*) \cong O^*$
- This manifold is also chiral, i.e. it admits no orientation-reversing self-diffeomorphism (like many other 3-manifolds)

$\mathbb{RP}^3 \# \mathbb{RP}^3$



Its fundamental group



$$\underbrace{\langle a, b \mid a^2 = 1 = b^2 \rangle}_{\mathbb{Z}_2 * \mathbb{Z}_2} = \underbrace{\langle a, c \mid a^2 = 1,\ aca^{-1} = c^{-1} \rangle}_{\mathbb{Z}_2 \ltimes \mathbb{Z}}, \quad c := ab$$

MCG and its u.i. representations

- The group of mapping classes is given by

$$\mathrm{MCG}_F \cong \mathrm{Aut}(\mathbb{Z}_2 * \mathbb{Z}_2) \cong \mathbb{Z}_2 * \mathbb{Z}_2 = \langle E, S \mid E^2, S^2 \rangle$$
$$E : (a, b) \to (b, a), \quad S : (a, b) \to (a, aba^{-1})$$

⇒ $ES + SE \subset$ centre of group algebra. Hence $\{1, E, S, ES\}$ generate algebra of irreducible representing operators.

⇒ Linear irreducible representations are at most 2-dimensional. They are: $E \mapsto \pm 1$, $S \mapsto \pm 1$ and, for $0 < \theta < \pi$,

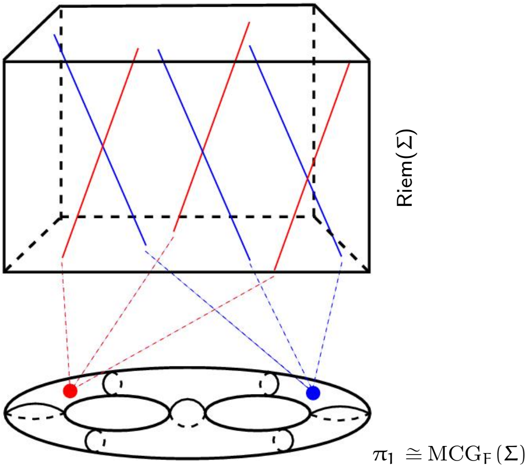
$$E \mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$
$$S \mapsto \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix},$$

⇒ There are two 'statistics sectors', which get 'mixed' by S; the 'mixing angle' is θ.

Prime Π	HC	S	C	N	$H_1(\Pi)$	$\pi_0(D_F(\Pi))$	$\pi_1(D_F(\Pi))$	$\pi_k(D_F(\Pi))$
S^3/D_8^*	+	+	+	−	$Z_2 \times Z_2$	O^*	0	$\pi_k(S^3)$
S^3/D_{8n}^*	+	+	+	−	$Z_2 \times Z_2$	D_{16n}^*	0	$\pi_k(S^3)$
$S^3/D_{4(2n+1)}^*$	+	+	+	+	Z_4	$D_{8(2n+1)}^*$	0	$\pi_k(S^3)$
S^3/T^*	?	+	+	−	Z_3	O^*	0	$\pi_k(S^3)$
S^3/O^*	w	+	+	+	Z_2	O^*	0	$\pi_k(S^3)$
S^3/I^*	?	+	+	−	0	I^*	0	$\pi_k(S^3)$
$S^3/D_8^* \times Z_p$	+	+	+	−	$Z_2 \times Z_{2p}$	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{8n}^* \times Z_p$	+	+	+	−	$Z_2 \times Z_{2p}$	$Z_2 \times D_{16n}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D_{4(2n+1)}^* \times Z_p$	+	+	+	+	Z_{4p}	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T^* \times Z_p$	?	+	+	−	Z_{3p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/O^* \times Z_p$	w	+	+	+	Z_{2p}	$Z_2 \times O^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/I^* \times Z_p$	?	+	+	−	Z_p	$Z_2 \times I^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/D'_{2^k(2n+1)} \times Z_p$	+	+	+	+	$Z_p \times Z_{2^k}$	$Z_2 \times D_{8(2n+1)}^*$	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$S^3/T'_{8\cdot 3^m} \times Z_p$	?	+	+	−	$Z_p \times Z_{3^m}$	O^*	Z	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_1)$	w	−	+	$(-)^p$	Z_p	Z_2	Z	$\pi_k(S^3)$
$L(p,q_2)$	$w+$	−	+	$(-)^p$	Z_p	$Z_2 \times Z_2$	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_3)$	w	−	−	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
$L(p,q_4)$	w	−	+	$(-)^p$	Z_p	Z_2	$Z \times Z$	$\pi_k(S^3) \times \pi_k(S^3)$
RP^3	+	−	−	+	Z_2	1	0	0
S^3	+	−	−	−	1	1	0	0
$S^2 \times S^1$	/	−	−	+	Z	$Z_2 \times Z_2$	Z	$\pi_k(S^3) \times \pi_k(S^2)$
R^3/G_1	/	+	−	+	$Z \times Z \times Z$	$\mathrm{St}(3,Z)$	0	$\pi_k(S^3)$
R^3/G_2	/	+	−	+	$Z \times Z_2 \times Z_2$	$\mathrm{Aut}_+^{Z_2}(G_2)$	0	$\pi_k(S^3)$
R^3/G_3	/	+	+	+	$Z \times Z_3$	$\mathrm{Aut}_+^{Z_2}(G_3)$	0	$\pi_k(S^3)$
R^3/G_4	/	+	+	−	$Z \times Z_2$	$\mathrm{Aut}_+^{Z_2}(G_4)$	0	$\pi_k(S^3)$
R^3/G_5	/	+	+	+	Z	$\mathrm{Aut}_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
R^3/G_6	/	+	+	−	$Z_4 \times Z_4$	$\mathrm{Aut}_+^{Z_2}(G_5)$	0	$\pi_k(S^3)$
$S^1 \times R_g$	/	+	−	−	$Z \times Z_{2g}$	$\mathrm{Aut}_+^{Z_2}(Z \times F_g)$	0	$\pi_k(S^3)$
$K(\pi,1)_{\mathrm{sl}}$	/	+	*	*	$A\pi$	$\mathrm{Aut}_+^{Z_2}(\pi)$	0	$\pi_k(S^3)$

taken from D.G. 1996

Superspace (Wheeler De Witt)



Final Remark: BH cosmology