Chapter Two: Relativistic accounts of the experiments of Trouton and Noble

2.0 Introduction: the 'Laue effect' and how to define it away

2.0.1 The fate of the experiments of Trouton and Noble in the history of the special theory of relativity. As we saw in chapter one, classical electrodynamics does predict the effect sought after in vain by Trouton and Noble (1903), i.e., a turning couple of the Coulomb forces acting on the plates of a moving condenser carrying a fixed charge. However, in Lorentz's version of the theory, this effect is exactly compensated by the turning couple coming from the intermolecular forces in the condenser, which Lorentz (1904b) assumed to transform in the exact same way as the Coulomb forces. Lorentz's theory predicts no such compensating mechanism for the effect Trouton (1902) was looking for in the experiment originally suggested to him by FitzGerald, i.e., an impulse upon charging or discharging a moving condenser. Instead, Lorentz attributed the negative result of this experiment to insufficient sensitivity of the apparatus. As I already pointed out in the introduction to part one, one needs the equivalence of mass and energy to understand how it can be, as is required by the principle of relativity, that the experiment will always yield a negative result, no matter how sensitive we make the apparatus.

To my knowledge, no one has ever pointed this out before, although the explanation seems quite straightforward. In fact, the Trouton experiment is remarkably similar to a thought experiment Einstein (1906) used in one of his derivations of $E = mc^2$. Moreover, Einstein's thought experiment was inspired by remarks of Poincaré (1900b) on the fate of Newton's reaction principle and the center of mass theorem in Lorentz's theory (Darrigol 1994b), and Lorentz himself would present Einstein's thought experiment in his lectures on relativity (Lorentz 1922, pp. 242–243). Yet, neither Einstein nor Lorentz seem to have thought of a connection with the Trouton experiment. Decades later, Laue would discuss this thought experiment in his contribution to the Schilpp volume dedicated to Einstein, in the same section in which he discusses the Trouton-Noble experiment (Laue 1949, pp. 524–528). Yet, Laue also failed to mention the Trouton experiment

Instead of living on in modern textbooks as a nice illustration of $E = mc^2$, the Trouton experiment seems to have been largely forgotten. I can think of several factors that may have contributed to this unhappy fate. First, recall that Larmor (1902; see section 1.1) essentially dismissed the basic idea of FitzGerald on which the experiment was based, a verdict Trouton accepted. And FitzGerald had meanwhile passed away. Both Larmor and Trouton clearly felt that the Trouton-Noble experiment was far more interesting than Trouton's original experiment. This is also the impression one gets from reading Lorentz's discussion of the two experiments in his classic paper of 1904 (Lorentz 1904b). Whereas the Trouton-Noble experiment plays an important role in the exposition of the theory, the discussion of the Trouton experiment is relegated to the very last section, almost as an appendix to the paper. To make matters worse, this section was omitted in the reprint of Lorentz's paper in the popular Teubner anthology on the principle of relativity (Blumenthal 1913[1]). Jakob Laub, one of Einstein's early collaborators, does not refer to Lorentz's important analysis in his brief discussion of the Trouton experiment in a review article on the experimental basis of special relativity (Laub 1910, pp. 428–430).[2] Laub lists the experiment as one of four first order electrodynamical (as opposed to optical) experiments with a negative result. Laub leaves it at that, presumably because the result is just what you would expect on the basis of the relativity principle. Laub does not address the problem of how it can be that the theories of Lorentz and Einstein predict different results for this experiment. I do not know of anyone before or since who has even raised this question.

The Trouton-Noble experiment, by comparison, has fared much better. On the experimental side, its null result was confirmed with greater accuracy in the 1920s by Tomashek (1926a, 1926b) and Chase (1926).[3] On the theoretical side, Max Laue in the course of developing his relativistic mechanics offered a general explanation of the experiment's negative result without simply appealing to the relativity principle (Laue 1911a, 1911b, 1911c, 1912a). Furthermore, in a paper devoted exclusively to the Trouton-Noble experiment (Laue 1912b), he supplemented this general and rather abstract explanation with a more intuitive account directly in terms of the forces acting on the condenser. The Trouton-Noble experiment, Laue emphasized, provides an example of an actual physical system exhibiting a peculiar effect Laue had found in developing his relativistic mechanics, an effect with no counterpart in Newtonian mechanics. As Laue liked to put it, giving the result a somewhat paradoxical flavor, *stressed systems in static equilibrium which are in uniform motion need a turning couple in order to sustain this motion*.[4] I will call this the *Laue effect* for short. It is most strikingly illustrated by a thought experiment involving a device known as the Lewis-Tolman bent lever (Lewis and Tolman 1909; Laue 1911c, 1912a).

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[1] Cf. Miller 1981, p. 391.

[2] I am grateful to John Stachel for bringing this paper to my attention. See Stachel et al. 1989, pp. 503–507, for discussion of the collaboration of Einstein and Laub.

[3] See Whittaker 1953, Vol. II, p. 29; Swenson 1972, pp. 147–148; Joos 1934, p. 448

[4] Laue 1911a, p. 149; 1911b, p. 168; 1911c, p. 514; 1912a, p. 163; 1912b, p. 169.

Pauli's celebrated review article contains a detailed discussion of Laue's account of the Trouton-Noble experiment (Pauli 1921, pp. 128–130). Discussions of the Trouton-Noble experiment, ranging from cursory and superficial to in-depth expositions of Laue's account, can be found in numerous textbooks on relativity and electrodynamics written since.[5]

2.0.2 The kinematical nature of the turning couples in the Trouton-Noble experiment. The main focus of this chapter will be on the Trouton-Noble experiment rather than on the Trouton experiment. Most of the chapter will be taken up by a detailed analysis of Laue's canonical account of the experiment. Taking my inspiration from Rohrlich (1960, 1965), I will argue that the Laue effect is an artifact of Laue's definitions of such quantities as the four-momentum and angular momentum of spatially extended open systems, definitions involving integration of the corresponding current densities over spacelike hyperplanes. By 'open' I mean that the system exchanges energy-momentum with its surroundings. This formally translates into the statement that the four-divergence of the energy-momentum tensor, the current density corresponding to four-momentum, does not vanish everywhere. One can prove that the integral of a system's energy-momentum tensor over a spacelike hyperplane is independent of which hyperplane we choose if and only if the system is closed (i.e., the divergence of its energy-momentum tensor vanishes everywhere). So, the definition of four-momentum for closed systems is unambiguous, but the definition of four-momentum for open systems necessarily involves a convention about how to choose the hyperplane over which we integrate the energy-momentum tensor. One has the same problem in defining the angular momentum for spatially extended systems. With the convention Laue uses, which is the convention still standard today, the angular momentum of static open systems is not conserved and as a consequence we find the Laue effect; with an alternative convention proposed by Rohrlich (1960, 1965), the angular momentum of static open systems is conserved and we do not find the Laue effect. This means that the Laue effect simply comes from choosing spacelike hyperplanes in Minkowski space-time in some particular way. The effect therefore tells us nothing about the detailed dynamics in the system, it is simply a reflection of the space-time

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[5] Discussions as thorough as those of Laue and Pauli are given by Becker (1962, I, pp. 394–401) and Tolman (1987, pp. 79–83). At the other end of the spectrum we find discussions that simply appeal to the relativity principle to explain the negative result of the Trouton-Noble experiment. Typically, the argument that is offered is simply that since the Coulomb forces do not make the condenser rotate in its rest frame, they cannot make it rotate in any other frame either. Examples in this category are: Lorrain and Corson (1970, p. 248), Rosser (1971, p. 86), Puri (1972, pp. 263–264), and Wangsness (1986, p. 498). In between these two extremes, we find discussions that mention that the turning couple of the electromagnetic forces is exactly compensated by a turning couple coming from the non-electromagnetic part of the system without going into great detail. Examples in this category are: Panofsky and Phillips (1955, p. 294; 1962, pp. 349–350), Kacser (1967, p. 216), Jánossy (1971, p. 66), and Schwartz (1977, pp. 190–191). I am grateful to Gordon Fleming for contributing several items to this list.

structure. In other words, the effect is purely kinematical, just as the basic relativistic phenomena of length contraction and time dilation.

What exactly is the Laue effect? Consider a system in static equilibrium. For a system to be static, its energy-momentum tensor has to satisfy two conditions. First, it has to be time independent in the system's rest frame. Second, those of its components that describe energy flow and (ordinary three-)momentum density have to vanish in the system's rest frame. For a system to be in static equilibrium it has to be static and closed. The system studied in the Trouton-Noble experiment is an example of a system in static equilibrium. The total system, which is static and closed, can be divided into an electromagnetic and a non-electromagnetic part, which both will be static open sub-systems. Suppose we have a system in static equilibrium such that the components of the energy-momentum tensor of one of its constituents describing the stresses on that constituent do not vanish in the system's rest frame. In a frame in which the system is moving, Laue found, these stresses give rise to momentum and this momentum will generally not be in the same direction as the system's velocity. As Lorentz already showed (see section 1.4, Eq. 1.38 and Eq. 1.62), this means that there will be a turning couple on the system. Obviously, this turning couple should somehow be compensated. Otherwise, we would have a violation of Einstein's principle of relativity. A closed static system, i.e., a system in static equilibrium, cannot be rotating in one frame and not be rotating in another.[6] The Laue effect is what prevents this from happening. A stressed system in static equilibrium which is in uniform motion will experience a turning couple that cancels the turning couple coming from the stresses.

Consider the Trouton-Noble condenser in its rest frame. The Coulomb force between the charged plates stresses the material part of the condenser. Without stresses of this kind the system could not possibly be in static equilibrium. The plates would simply collapse onto one another. In a frame in which the condenser is in uniform motion these stresses give rise to a turning couple. This turning couple is canceled by the turning couple coming from the Coulomb forces evaluated in this frame.

The Coulomb forces themselves can also be described in terms of stresses, called electromagnetic stresses or Maxwell stresses. The turning couple coming from the Coulomb forces is due to these Maxwell stresses in exactly the same way as the turning couple it opposes is due to the stresses in the material part of the condenser. Moreover, the fact that the two turning couples cancel one another in the frame in which the condenser is moving is directly

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[6] Somewhat surprisingly, perhaps, a closed non-static system *can* be rotating in one frame but not in another. In section 2.4, I will show how we can understand this on the basis of the relativity of simultaneity. I am grateful to John Norton for clarifying this point.

related to the fact that the electromagnetic stresses and the non-electromagnetic stresses cancel one another in the system's rest frame (otherwise the system could not be in static equilibrium).

At first sight, Laue's relativistic account of the Trouton-Noble experiment strongly resembles Lorentz's. According to both, there is a turning couple coming from the electromagnetic part of the system and a turning couple coming from the non-electromagnetic part of the system and the two of them exactly cancel one another. Moreover, both for Lorentz and for Laue, this cancellation of turning couples is directly related to the fact that the system is in static equilibrium.

There is an important difference though, as Laue emphasized. Whereas Lorentz ascribed this non-Newtonian effect—simply setting a system in motion does not call forth any turning couples in classical mechanics!—to peculiarities of electromagnetic and molecular forces, Laue saw it as a manifestation of a general effect of a new relativistic mechanics. That, of course, is why I named the effect after Laue and not after Lorentz. After giving his interpretation of Lorentz's account of the Trouton-Noble experiment in his first published discussion of the problem (see section 1.4), Laue writes: "Although this answer undoubtedly hits upon the right solution, it is not quite satisfactory, in that it takes recourse to molecular theory, with which the problem in and of itself has nothing to do" (Laue 1911a, p. 136). He concludes another exposition of the Laue effect and the Trouton-Noble experiment by saying: "In this sense, the Trouton-Noble experiment decides in favor of the dynamics of relativity theory and against Newtonian mechanics" (Laue 1912a, p. 164).

As I mentioned above, it depends on one's definitions of four-momentum and angular momentum whether one has the Laue effect or not. Under Rohrlich's definition, relativistic mechanics, like classical mechanics and contrary to what Laue thought, predicts no such thing as delicately balanced turning couples acting on stressed systems in static equilibrium when they are in motion. Stresses in one frame do not give momentum in other frames and momentum is always in the direction of the velocity.

In the end, these observations only reinforce the general point Laue makes in the two quotations I gave above. What Laue says is basically that the Laue effect is part of what I called, following Planck, the general dynamics. My analysis shows that we can be more specific. It is part of what I called, following John Stachel's suggestion, the general dynamical consequences of the kinematics (see the introduction to part one). In other words, the Laue effect is purely kinematical. It is simply a manifestation of the normal spatio-temporal behavior of stressed systems in static equilibrium in Minkowski space-time.

With the help of Laue's analysis of the Trouton-Noble experiment, it is easy to produce a relativistic account of the Trouton experiment. In fact, I already gave the essentials of such an account in the introduction to part one. As Lorentz pointed out, a moving condenser gains (loses) momentum when it is (dis-)charged. It does not follow, however, that the condenser will experience a jolt whenever it is charged or discharged, as Lorentz thought on the basis of Newtonian conservation of momentum. In relativistic mechanics, a change in momentum can be accompanied by a change in velocity, but also by a change in mass, a possibility not allowed for in Newtonian mechanics. Consequently, the relativistic momentum balance for the condenser in the Trouton experiment, evaluated in a frame in which the condenser is moving, is as follows. When the condenser is charged, it gains energy, hence mass, hence momentum, while the source supplying the energy to charge the condenser loses the same amount of energy, mass, and momentum. When the condenser is discharged, it loses energy, mass, and momentum to whatever system outside is receiving it. The Trouton experiment thus nicely illustrates the most famous result of relativistic mechanics, the equivalence of mass and energy.[7]

Notice that the transfer of four-momentum from the battery to the condenser in the Trouton experiment, unlike the Laue effect in the Trouton-Noble experiment, can not be defined away. This immediately tells us that it is not a purely kinematical effect. To be sure, the effect reflects a very general property of relativistic systems and is independent of the specific nature of the system. But more is involved than just the system's spatio-temporal behavior. Its status is thus the hybrid status accorded to the Laue effect in the remarks by Laue I quoted above.

2.0.3 Outline of chapter two. In sections 2.1 through 2.6, I will provide further details on the relativistic arguments outlined above. All calculations will be done with the help of the energy-momentum tensor. Building on work by Einstein, Lorentz, Planck (1906a, 1907, 1908), Abraham (1910), Sommerfeld (1910a, 1910b), and, especially, Minkowski (1908),[8] Laue put this quantity, originally defined in the context of electrodynamics only, at the center of his relativistic mechanics.[9] In section 2.1, I will introduce the energy-momentum tensor and go over an illuminating proof of the theorem that forms the core of a modern relativistic account of the Trouton-Noble experiment. Under its standard definition (i.e., as the space integral of the $\mu 0$-components of the energy-momentum tensor) four-momentum transforms as a four-vector if and only if the four-divergence of the corresponding energy-momentum tensor vanishes everywhere. In other words, four-momentum transforms as a four-vector if and only if the

[7] Since energy is a component of four-momentum, the details of this account of the Trouton experiment will depend on whether we use Laue's or Rohrlich's definition of that quantity (see section 2.5).

[8] The papers cited here are the relevant papers cited in Laue 1911a. I am grateful to Rita Fountain-Lübke for providing me with the exact reference to Abraham 1910.

[9] Laue 1911a, p. 140. I do not know of any comprehensive study in the history of science literature devoted to these important developments, a very unfortunate situation that I hope will be remedied in the near future by John Stachel's work on the history of relativity theory. There is a brief discussion in Miller 1981, pp. 367–374. Norton 1992a, pp. 42–53, also contains some useful remarks. By far the best discussion is still Laue's contribution to the Schilpp volume on Einstein (Laue 1949).

system is closed. Laue's account of the Trouton-Noble experiment (Laue 1911a) is based on his claim that the energy and momentum of what he called a "complete static system" transform as the components of a four-vector. Substituting 'four-momentum under the standard definition' for 'energy and momentum' and 'closed' for 'complete,' one will recognize Laue's claim as a weaker version of the theorem mentioned above. Laue uses the unnecessary extra condition that the system is static.

As I will show in detail, Laue's imposing of this overly restrictive condition is related to an error in his proof of his claim. This error is easily corrected by invoking the assumption that the system is static at one more juncture in the proof. Since the systems Laue is interested in are all static, it does not really matter whether we prove the result for complete systems or for complete static systems. However, it will be through tracing Laue's error that we naturally end up identifying the arbitrary element in his definition of four-momentum for spatially extended open systems, i.e., the particular convention Laue uses for choosing spacelike hyperplanes over which to integrate the energy-momentum tensor.

In section 2.2, I will give a detailed analysis of this arbitrariness. The exact same problem lies at the root of the infamous spurious factor $^4\!/_3$ in the mass-energy relation for Lorentz's purely electromagnetic model of the electron. It is not surprising that this problem is related to the Trouton-Noble experiment. Laue, in fact, applies his claim about complete static systems to two special cases, the Trouton-Noble experiment and the electron model of Lorentz amended by Poincaré. I will briefly go over the history of the $^4\!/_3$-puzzle and show how the issue was definitively settled by Rohrlich (1960, 1965). Rohrlich's solution suggests an alternative account of the Trouton-Noble experiment, based on an alternative convention for choosing the spacelike hyperplanes in the relevant definitions. Instead of the delicately balanced turning couples of Laue's account, this alternative account does not involve any turning couples whatsoever. The most concise and relativistically kosher way of comparing the two accounts of the experiments is to give the argument in terms of angular momentum, which is what I will do at the end of section 2.2. The upshot of this comparison will be that the Laue effect is an artifact of an arbitrariness in the definition Laue uses for quantities such as four-momentum and angular momentum for spatially extended open systems.

In section 2.3, I will work out the details of Laue's 1911 'four-momentum'-account of what happens in the electromagnetic and the non-electromagnetic parts of the condenser in the Trouton-Noble experiment and contrast it with the 'four-momentum'-account suggested by Rohrlich's work. Recall that, in the analysis of both Lorentz and Laue, a turning couple is found whenever the direction of the momentum of some part of the system is different from the direction of the system's velocity. Under Rohrlich's definition, four-momentum always has the same direction as four-velocity. This gives us another way of understanding the absence of the

Laue effect in the Rohrlich picture, and hence another way of seeing that the effect is an artifact of arbitrary definitions.

Since the charge distribution on our box-shaped condenser has far less symmetry than the spherical charge distribution constituting Lorentz's purely electromagnetic electron, the case of the Trouton-Noble experiment gives a much richer harvest than the case of the electron in terms of seemingly dynamical effects that on closer examination turn out to be purely kinematical. In the Laue picture of what happens in a moving condenser, we not only have delicately balanced turning couples, but also exchanges of four-momentum and angular momentum between the electromagnetic and the non-electromagnetic parts of the condenser if the whole system were slowly being rotated (slowly so as to preserve the static character of the situation). In addition, even when the system is in uniform motion with no rotation at all, there will be a steady flow of angular momentum from the electromagnetic to the non-electromagnetic part, or vice versa, depending on the angle between the plates and their velocity. In the Rohrlich picture, we have none of these phenomena. The Rohrlich picture fits better with our intuitions about what it means for a system to be in static equilibrium, I think, than the Laue picture. I want to emphasize, however, that I do *not* want to argue that the definitions on which Rohrlich's picture is based are superior to the definitions on which Laue's picture is based. Both types of definitions have pros and cons. What I do want to argue is the following. If an effect is present under one definition and absent under another, differing from the first only in its arbitrary conventions about picking spacelike hyperplanes, then the effect is purely kinematical. At the end of section 2.3, I will give a more detailed and careful version of the argument establishing the purely kinematical nature of the Laue effect.

In section 2.4, I will present a slightly idealized version of Laue's more intuitive 'forces'-account (Laue 1912b).[10,11] I will also take a closer look at the expressions for the various contributions to the energy of the system in the Laue picture. Adapting an argument due to Einstein (1907b), I will show how the different terms can be interpreted directly in terms of the relativity of simultaneity. This is tantamount to recognizing that these expressions simply reflect a particular choice of a hyperplane of simultaneity. This means that we can credit Einstein with a deeper understanding of situations such as the one encountered in the Trouton-Noble experiment than Laue. A clear understanding of the role of the relativity of simultaneity in the Trouton-Noble experiment will also enable us to get a more intuitive grasp of how something which sounds as dynamical as a turning couple can actually be a purely kinematical effect.

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[10] In section 1.2, I already gave an overly simplified version of this account.

[11] This calculation in the Laue picture does not seem to have a natural analogue in the Rohrlich picture, although I have to confess that I have not tried very hard to find it. For my purposes, it was sufficient to develop the analogues in the Rohrlich picture of the calculations in the Laue picture in sections 2.2 and 2.3.

Finally, in section 2.5, I will discuss the relation between the Trouton-Noble experiment and the Lewis-Tolman bent lever, and the relation between the Trouton experiment and $E = mc^2$.

2.1 The transformation of four-momentum for 'complete static systems'

2.1.1 The energy-momentum tensor. Historically, the energy-momentum tensor was first encountered in electrodynamics. Various quantities that were introduced in section 1.4—the electromagnetic energy density u, the electromagnetic momentum density $\mathbf{g}$, and the Maxwell stress tensor T_{ij}—can be combined in the electromagnetic energy-momentum tensor $T^{\mu\nu}$. In an arbitrary Lorentz frame on Minkowski space-time, the contravariant symmetric electromagnetic energy-momentum tensor is defined as (summation over repeated indices being understood):[12]

$$T^{\mu\nu}_{\text{EM}} = \mu_0^{-1}\left(F^{\mu}{}_{\alpha}\, F^{\alpha\nu} + \tfrac{1}{4}\eta^{\mu\nu}\, F_{\alpha\beta}\, F^{\alpha\beta}\right), \tag{2.1}$$

where $F^{\mu\nu} \equiv \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$ is the contravariant electromagnetic field tensor. The Greek indices run from 0 to 3, corresponding to the space-time coordinates (ct, x, y, z). Derivatives with respect to these coordinates are written as ∂_{μ}. Indices are raised and lowered with the help of the Minkowski metric $\eta^{\mu\nu} \equiv \text{diag}(1,-1,-1,-1)$. The four-vector potential is defined as $A^{\mu} \equiv (\phi/c,\mathbf{A})$, where ϕ is the ordinary scalar potential and $\mathbf{A}$ is the three-vector potential. With the help of the relation $\mathbf{E} = -\text{ grad } \phi - \partial\mathbf{A}/\partial t$, the relation $\mathbf{B} = \text{curl } \mathbf{A}$, and the definitions above, it is easily verified that the field tensor $F^{\mu\nu}$ can be represented by the following matrix:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{pmatrix}. \tag{2.2}$$

The quantities $F^{\mu}{}_{\nu} = F^{\mu\rho}\,\eta_{\rho\nu}$ and $F_{\mu\nu} = \eta_{\mu\rho}\,F^{\rho\sigma}\,\eta_{\sigma\nu}$ can be represented by similar matrices. If we change the signs in columns 1 through 3 in Eq. 2.2, we obtain the matrix representing $F^{\mu}{}_{\nu}$; if we change the signs in column 0 and row 0 in Eq. 2.2, we obtain the matrix representing $F_{\mu\nu}$. With the help of these matrices one can read off expressions for the various components of $T^{\mu\nu}_{\text{EM}}$ in Eq. 2.1. The 00-component turns out to be equal to the energy density u (cf. Eq. 1.68):

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[12] See, e.g., Pauli 1921, pp. 85–86; Jackson 1975, pp. 601–606. Unlike Pauli or Jackson, I use SI or MKSA units. A very thorough discussion of how to define the canonical energy-momentum tensor (or momentum current) as the Noether current (and four-momentum as the corresponding Noether charge) associated with translation invariance of the Lagrangian, and of how to get from the canonical energy-momentum tensor to the symmetric energy-momentum tensor such as the one given in Eq. 2.1 can be found in Soper 1976, pp. 101-123.

$$T^{00}_{\mathrm{EM}} = \frac{1}{2}\,\varepsilon_0\,E^2 \;+\; \frac{1}{2}\,\mu_0^{-1}\,B^2 \equiv u\;. \tag{2.3}$$

The components (10, 20, 30)—and, likewise, the components (01, 02, 03)—turn out to be equal to c times the momentum density **g** (cf. Eq. 1.35):

$$\left(T^{10},\; T^{20},\; T^{30}\right)_{\mathrm{EM}} = c\left(\varepsilon_0\,\mathbf{E}\times\mathbf{B}\right) \equiv c\,\mathbf{g}\;. \tag{2.4}$$

Finally, the ij-components (with i and j running from 1 to 3) turn out to be equal to minus the Maxwell stress tensor (cf. Eq. 1.53):

$$T^{ij}_{\mathrm{EM}} = -\left\{\varepsilon_0\left(E^iE^k - \tfrac{1}{2}\delta^{ik}E^2\right) + \mu_0^{-1}\left(B^iB^k - \tfrac{1}{2}\delta^{ik}B^2\right)\right\}. \tag{2.5}$$

The interpretation of the various components of the energy-momentum tensor is the same independently of the nature of the system under consideration. The 00-component always represents the energy density, the i0-components (divided by c) represent the components of the momentum density, the 0i-components (times c) represent the components of the energy current density, and the ij-components represent the stresses, or, equivalently, the components of the various momentum current densities. Notice that the interpretation of the *i0*-components and the *0i*-components incorporates the result that an energy flow always corresponds to momentum. In the electromagnetic case, this is expressed in terms of the Poynting vector $\mathbf{S} \equiv \mu_0^{-1}\,\mathbf{E}\times\mathbf{B}$ (see, e.g., Jackson 1975, p. 237), representing the energy current density, and the electromagnetic momentum density $\mathbf{g} \equiv \varepsilon_0\,\mathbf{E}\times\mathbf{B}$. The relation between these two quantities is: $\mathbf{g} = \varepsilon_0\,\mu_0\,\mathbf{S} = \mathbf{S}/c^2$.[13]

Laue (1911a, p. 141) cites Planck 1908 for this result.[14] In section 1.4, I already gave an extensive quotation from this paper of Planck, discussing the complication of the general concept of momentum—or "quantity of motion" as it is called in Planck 1908—brought about by Abraham's introduction of electromagnetic momentum in 1903. Immediately after the passage I quoted in section 1.4, Planck writes:

> Now, is it possible, in the context of general dynamics, to define the quantity of motion in the same uniform manner as it used to be defined in mechanics, even though it now comprises both the mechanical and the electromagnetic kind? An affirmative answer to this question would definitely lead to an advance in our understanding of the real meaning of the reaction

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[13] As Laue emphasizes, this result is elegantly encoded in the symmetry of the energy-momentum tensor, a property which can be motivated by insisting on angular momentum conservation in relativistic mechanics (Laue 1911a, pp. 140–141, pp. 147–150, and Eqs. 2.44–2.47 below). Laue must have been pleased to find that the integrals over the energy-momentum tensor representing four-momentum in different Lorentz frames likewise encoded what I call the Laue effect.

[14] It can also be found in the first edition (of 1909) of Lorentz's 1906 lectures in New York (Lorentz 1916, p. 33).

> principle [by which, as is explained at the beginning of the paper, Planck means momentum conservation].
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> Such a uniform definition of the quantity of motion indeed appears to be possible and feasible, at least if, at the same time, the validity of the Einsteinian theory of relativity is admitted. (Planck 1908, p. 216)

After a brief and obligatory cautionary remark on this last assumption, Planck, in the remainder of this short paper, elaborates on this uniform definition of momentum, describing, in effect, the energy-momentum tensor and the physical interpretation of its components. The crucial sentences read:

> In relativity theory, one can quite generally reduce the quantity of motion to the vector representing the energy flow [...] This vector divided by the square of the velocity of light is quite generally the quantity of motion per unit volume [...]
>
> From the point of view sketched here, the principle of the equality of action and reaction [i.e., momentum conservation] can quite generally be called the "theorem of the inertia of energy."
>
> We can even go one step further. Just as the constancy [read: conservation] of energy implies the notion of the energy flow, so does the constancy [read: conservation] of the quantity of motion necessarily imply the notion of the "flow of the quantity of motion," or, put more concisely, the "momentum flow." [...] However, an essential difference compared to the energy flow is that energy is a scalar, whereas the quantity of motion is a vector. Consequently, [...] the momentum flow at a certain position is a tensor triple [...] characterized by six components. (Ibid., pp. 217–218)

2.1.2 Laue's notion of a 'complete static system.' Laue's account of the Trouton-Noble experiment is based upon a general claim about the transformation behavior of the energy and momentum of what he called a "complete static system" (*vollständiges statisches System*,[15] Laue 1911a, section 5, pp. 150–153). By "complete" Laue roughly means that the system does not interact with anything but itself. Laue is rather vague in his explanatory prose,[16] but from the discussion earlier on in his paper (especially, ibid., p. 142, Eq. (5)) and from the proof he offers for his general claim, it is clear that a system being *complete* means that its total energy-momentum tensor is divergence free

$$\partial_\mu T^{\mu\nu}_{\rm tot} = 0, \qquad \text{[complete/closed]} \qquad (2.6)$$

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[15] Miller (1981, p. 373) inaccurately translates this as "perfectly static system." If we read 'perfectly static' as 'in static equilibrium,' no real harm is done, since the latter is equivalent to 'static and complete.' A better term would be 'closed static system' or 'isolated static system.' 'Complete static system,' however, is closest to the German original. It is also the translation used by Norton (1992a, p. 51). I will consistently use the term 'open' for the opposite of 'complete.' The reason I want to avoid the term 'incomplete' is that 'open' has nothing to do with 'incomplete' in the sense in which that term is now routinely used in the philosophy of quantum mechanics. Tolman (1987, pp. 81–83) uses the expression 'complete static system' in the same way as Laue. I suspect he took the phrase from Laue, although he does not cite Laue (or anyone else) for it.

[16] Cf. Norton 1992a, p. 51.

where $T^{\mu\nu}_{\text{tot}}$ is the sum of the energy-momentum tensors for all constituents of the system. We will be interested only in the special case where the system consists of two parts, a purely electromagnetic part and a part involving at least some non-electromagnetic stuff. For convenience, I will call these two parts the electromagnetic and the non-electromagnetic part, respectively. So, I will assume that $T^{\mu\nu}_{\text{tot}}$ is of the form:

$$T^{\mu\nu}_{\text{tot}} = T^{\mu\nu}_{\text{EM}} + T^{\mu\nu}_{\text{non-EM}}. \tag{2.7}$$

The conditions expressing that a system is *static* are

$$\partial'_0 T'^{\mu\nu} = 0, \qquad T'^{i0} = T'^{0i} = 0, \qquad \text{[static]} \tag{2.8}$$

where the primes refer to the system's rest frame, and where $T'^{\mu\nu}$ can refer to any of the three energy-momentum tensors in Eq. 2.7.

Laue makes the following claim. If a system is a complete static system, then the total energy U and (c times) the total momentum $\mathbf{P}$ of the system, defined as

$$U \equiv \int T^{00}_{\text{tot}} d^3x \ , \quad cP^i \equiv \int T^{i0}_{\text{tot}} d^3x \ , \tag{2.9}$$

transform as a four-vector under Lorentz transformation.

A modern reader will recognize this claim as a weaker version of the following well-known result. If some energy-momentum tensor $T^{\mu\nu}$ is divergence free (and falls off fast enough as we go to spatial infinity, a fairly weak condition satisfied by all systems we will consider), then the four-momentum P^{μ}, defined as

$$P^{\mu}(t) \equiv (U(t),\ c\mathbf{P}(t)) \equiv \int T^{\mu 0}_{\text{tot}}(t, \mathbf{x}) d^3x \ , \tag{2.10}$$

is conserved and transforms as a four-vector under Lorentz transformation. Notice that the system does not have to be static. I will go over an illuminating proof of this stronger result (and its converse), a proof due to Rohrlich (1965). The purpose of this exercise is twofold.

First, it will enable us to bring out an error in Laue's proof of the weaker result, and to understand why, given this error, he introduced the unnecessary extra assumption that the system be static. Basically, Laue's error is that he sets space integrals in one inertial frame equal to space integrals in others (Laue 1911a, p. 144). As we will see below, this move is allowed if the system is static, but in general it is not. This should not be surprising. As Rohrlich emphasizes, a space integral in one frame, i.e., an integral over a hyperplane of simultaneity in that frame, will remain an integral over that *same* hyperplane under any Lorentz transformation.

If we transform to a frame that is moving with respect to the original one, this hyperplane will no longer be a hyperplane of simultaneity in the new frame. Hence, the integral will not be a space integral in the new frame.

Since Laue is interested only in static systems, it would be a little pedantic to rehearse Rohrlich's modern proof for the sole purpose of bringing out this seemingly inconsequential error of Laue in 1911. Obviously, I have another reason for presenting Rohrlich's proof. It naturally brings out an arbitrariness in the standard definition of four-momentum that Laue uses in which four-momentum is an integral over hyperplanes of simultaneity in whatever Lorentz frame we happen to be using. Rohrlich proposed an alternative definition in which four-momentum is an integral over a fixed hyperplane, viz. the hyperplane of simultaneity in the system's rest frame. This in turn will show, as I mentioned in the introduction to this chapter, that the turning couples in the Laue picture of what happens in the Trouton-Noble experiment are artifacts of the definition of four-momentum Laue uses.

Having spelled out these objectives, let me now give the theorem and its proof.[17]

2.1.3 A general theorem: the four-momentum of a system transforms as a four-vector if and only if the system is closed. I need to make one more preliminary remark. Throughout the rest of the chapter, I assume the energy-momentum tensor falls off faster than $1/r^3$ as we go to spatial infinity

$$T^{\mu\nu} \rightarrow 0 \;\text{ faster than }\; \frac{1}{r^3} \;\text{ for }\; r \equiv \sqrt{x^2 + y^2 + z^2} \rightarrow \infty. \tag{2.11}$$

This condition is satisfied by all systems we will consider. At great distances from the condenser in the Trouton-Noble experiment, its electromagnetic field will be the field of a dipole. The energy-momentum tensor of the field of the condenser will therefore drop off as $1/r^6$ (see, e.g., Jackson 1975, p. 138).

With the condition in Eq. 2.11 in place, I will prove that four-momentum, in its standard definition (see Eq. 2.10), is conserved and transforms as a four-vector under Lorentz transformation if and only if the corresponding energy-momentum tensor is divergence free. Schematically, this theorem can be stated as:[18]

---

[17] I want to emphasize that I do not intend to give a mathematically rigorous proof of this theorem. Readers with low tolerance for mathematical sloppiness are advised to skip the next subsection altogether and work out the proof for themselves with the help, for instance, of Hawking and Ellis 1973, p. 62. My proof caters to those readers who are more interested in acquiring a good intuitive understanding of the theorem than in mathematical niceties.

[18] The expression "P^μ is a conserved four-vector" is sloppy. It suggests that P^μ is a local vector field assigning to each point P of the space-time manifold an element of the tangent space at P. In fact, P^μ is a highly non-local object involving integration over a spacelike hyperplane in space-time. P^μ is not a vector, it only

$$P^{\mu} \text{ is a conserved four-vector} \quad \Leftrightarrow \quad \partial_{\nu} T^{\mu\nu} = 0 \text{ everywhere.} \tag{2.12}$$

Following Rohrlich (1965, pp. 89–90, and appendix A1-5, pp. 279–281), I will prove this theorem by proving a far more general theorem of which it essentially is a special case.

Consider an arbitrary well-behaved[19] spacelike hypersurface Σ in Minkowski space-time, stretching out to spatial infinity, and some arbitrary tensor field $A^{\mu\nu\ldots\kappa\lambda}$ that vanishes at spatial infinity. Define the object $A_{\Sigma}^{\mu\nu\cdot\cdot\cdot\kappa}$, which transforms as a tensor:[20]

$$A_{\Sigma}^{\mu\nu\cdot\cdot\cdot\kappa} \equiv \int_{\Sigma} A^{\mu\nu\cdot\cdot\cdot\kappa\lambda} \, d\Sigma_{\lambda}, \tag{2.13}$$

with $d\Sigma^{\lambda} = n^{\lambda}\, d\Sigma$, where n^{λ} is a timelike vector field giving the normal on the hypersurface Σ in the future time direction and $d\Sigma$ is an invariant hypersurface element. I will prove that $A_{\Sigma}^{\mu\nu\cdot\cdot\cdot\kappa}$ is independent of our choice of Σ if and only if $\partial_{\lambda} A^{\mu\nu\ldots\kappa\lambda} = 0$ everywhere. Schematically, this theorem can be stated as:

$$A_{\Sigma}^{\mu\nu\cdot\cdot\cdot\kappa} \text{ is independent of } \Sigma \quad \Leftrightarrow \quad \partial_{\lambda} A^{\mu\nu\cdot\cdot\cdot\kappa\lambda} = 0 \text{ everywhere.} \tag{2.14}$$

The proof of the theorem is based on an application of the obvious generalization of Gauss's well-known theorem from three to four dimensions.[21]

Consider two arbitrary well-behaved spacelike hypersurfaces,[22] Σ^1 and Σ^2, stretching out to infinity, as illustrated for a 2-dimensional Minkowski space-time in Fig. 2.1. Let Σ_{top}^{1} be the union of those segments $\Sigma_{i}^{1} \subset \Sigma^{1}$ where Σ^1 lies to the future of Σ^2, and let $\Sigma_{\text{bottom}}^{1}$ be the union of those segments $\Sigma_{i}^{1} \subset \Sigma^{1}$ where Σ^2 lies to the future of Σ^1; Σ^2 can likewise be divided into Σ_{top}^{2} and $\Sigma_{\text{bottom}}^{2}$. Let Σ^{∞} be a timelike hyperplane connecting Σ^1 and Σ^2 in the limit of $r \rightarrow \infty$. Let V_1 be the total space-time volume enclosed by $\Sigma_{\text{top}}^{1}, \Sigma_{\text{bottom}}^{2}$, and Σ^{∞}; and let V_2 be the total space-time volume enclosed between $\Sigma_{\text{top}}^{2}, \Sigma_{\text{bottom}}^{1}$, and Σ^{∞}.

---

transforms as one. And even that statement needs to be qualified. P^{μ} transforms as a four-vector only under global Lorentz transformations. Given the non-local character of P^{μ}, it makes no sense to consider its transformation under local coordinate transformations. In summary, the phrase "P^{μ} is a conserved four-vector" should be read as shorthand for "P^{μ} is conserved and transforms as a four-vector under global Lorentz transformations."

19 I will not bother to spell out what I mean by well-behaved.

20 Because of its non-local character, $A_{\Sigma}^{\mu\nu\cdot\cdot\cdot\kappa}$ is not what we usually call a tensor, it only transforms as one.

21 I will not bother to prove either the original theorem or its generalization.

22 Rohrlich restricts himself to arbitrary hyperplanes, the only hypersurfaces we will actually use in our applications of Eq. 2.14, but nothing in his proof hinges on the hypersurfaces being of this special kind.

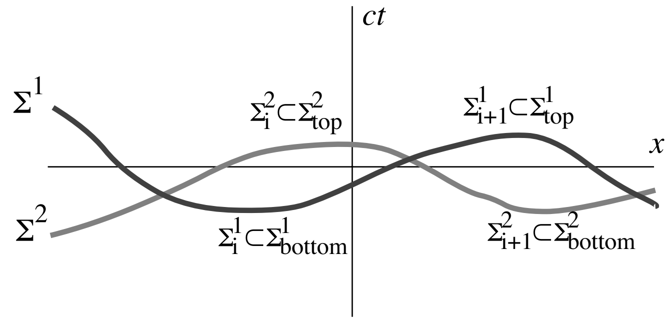

Figure 2.1 The hypersurfaces Σ^1 and Σ^2.

Consider the integral of $A^{\mu\nu\ldots\kappa\lambda}$ over the collection of closed hypersurfaces made up by the union $\Sigma^1 \cup \Sigma^2 \cup \Sigma^\infty$. According to the generalization of Gauss's theorem to four dimensions, this integral is equal to the integral of $\partial_\lambda A^{\mu\nu\ldots\kappa\lambda}$ over the space-time volume $V_1 \cup V_2$. Because $d\Sigma^\lambda$ in Eq. 2.13 always points in the positive time direction, whereas in the integral over $\Sigma^1 \cup \Sigma^2 \cup \Sigma^\infty$ we need the normal in the negative time direction on Σ^1_{bottom} and Σ^2_{bottom}, it is convenient to break up the integral over $\Sigma^1 \cup \Sigma^2 \cup \Sigma^\infty$ into an integral over $\Sigma^1_{\text{top}} \cup \Sigma^2_{\text{bottom}}$ and an integral over $\Sigma^2_{\text{top}} \cup \Sigma^1_{\text{bottom}}$. Since we are assuming that $A^{\mu\nu\ldots\kappa\lambda} \to 0$ faster than $1/r^3$ as $r \to \infty$, i.e., faster than the hypersurface area of Σ^∞ goes to infinity in this limit, the integral over Σ^∞ vanishes. So, we can write:

$$\begin{aligned}
&\int_{\Sigma^1_{\text{top}}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^1_\lambda - \int_{\Sigma^2_{\text{bottom}}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^2_\lambda = \int_{V_1} \partial_\lambda A^{\mu\nu\cdots\kappa\lambda}\, dV, \\
&\int_{\Sigma^2_{\text{top}}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^2_\lambda - \int_{\Sigma^1_{\text{bottom}}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^1_\lambda = \int_{V_2} \partial_\lambda A^{\mu\nu\cdots\kappa\lambda}\, dV.
\end{aligned} \tag{2.15}$$

Subtracting the second equation from the first, we find that

$$\int_{\Sigma^1 = \Sigma^1_{\text{top}} \cup \Sigma^1_{\text{bottom}}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^1_\lambda - \int_{\Sigma^2 = \Sigma^2_{\text{bottom}} \cup \Sigma^2_{\text{top}}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^2_\lambda = \int_{V_1} \partial_\lambda A^{\mu\nu\cdots\kappa\lambda}\, dV - \int_{V_2} \partial_\lambda A^{\mu\nu\cdots\kappa\lambda}\, dV. \tag{2.16}$$

With the help of this relation we can prove the '$\Leftarrow$'-implication in Eq. 2.14. On the assumption that $\partial_\lambda A^{\mu\nu\ldots\kappa\lambda} = 0$ everywhere, the right hand side of Eq. 2.16 vanishes. Therefore,

$$A_{\Sigma^1}^{\mu\nu\cdots\kappa} = A_{\Sigma^2}^{\mu\nu\cdots\kappa}. \tag{2.17}$$

Since Σ^1 and Σ^2 were chosen arbitrarily, this proves the 'if'-part of the theorem in Eq. 2.14.

Adding the two equations in Eq. 2.15, we find that

$$\int_{\Sigma' \equiv \Sigma^1_{top} \cup \Sigma^2_{top}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^1_\lambda - \int_{\Sigma'' \equiv \Sigma^1_{bottom} \cup \Sigma^2_{bottom}} A^{\mu\nu\cdots\kappa\lambda}\, d\Sigma^2_\lambda = \int_{V_1 \cup V_2} \partial_\lambda A^{\mu\nu\cdots\kappa\lambda}\, dV. \tag{2.18}$$

With the help of this relation we can prove the '⇒'-implication in Eq. 2.14. On the assumption that $A_\Sigma^{\mu\nu\cdots\kappa}$ is independent of Σ, $A_{\Sigma'}^{\mu\nu\cdots\kappa} = A_{\Sigma''}^{\mu\nu\cdots\kappa}$ and the left hand side of Eq. 2.18 vanishes. By choosing suitable hypersurfaces Σ' and Σ'', $V_1 \cup V_2$ on the right hand side can be made to coincide with any arbitrary space-time volume. For the integral to vanish for arbitrary volumes, it has to be the case that the integrand vanishes everywhere. This proves the 'only if'-part of the theorem in Eq. 2.14.

Consider the special case where for $A^{\mu\nu\ldots\kappa\lambda}$ we take the energy-momentum tensor $T^{\mu\nu}$, and for the hypersurface Σ we take a hyperplane $\Sigma(n^\mu, \tau)$, defined by the equation (cf. Rohrlich 1965, p. 279, Eq. (A1-55))

$$n_\mu x^\mu = -c\tau. \tag{2.19}$$

Suppose in a Lorentz frame with coordinates x'^μ, the normal takes the simple form $n'^\mu = (1, 0, 0, 0)$. This means that, in the x'^μ-frame, $\Sigma(n^\mu, \tau)$ is a hyperplane of simultaneity at time $t' = \tau$. All this is illustrated in Fig. 2.2.

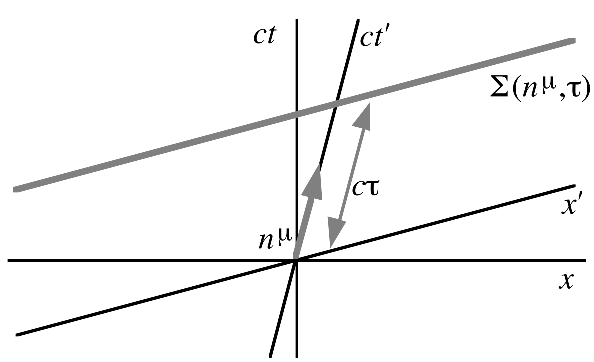

Figure 2.2 The hyperplane $\Sigma(n^\mu, \tau)$.

Define (cf. the definition of $A_\Sigma^{\mu\nu\cdots\kappa}$ in Eq. 2.13):

$$T^{\mu}_{\Sigma(n^{\mu},\tau)} \equiv \int_{\Sigma(n^{\mu},\tau)} T^{\mu\nu} n_{\nu} d\Sigma . \tag{2.20}$$

In the x'^{μ}-frame, $T^{\mu}_{\Sigma(n^{\mu},\tau)}$ becomes an ordinary space integral which is equal to P'^{μ} under its standard definition (see Eq. 2.10):

$$T'^{\mu}_{\Sigma(n^{\mu},\tau)} = \int_{\Sigma(n^{\mu},\tau)} T'^{\mu\nu} n'_{\nu} d\Sigma = \int T'^{\mu 0}(t', \mathbf{x}')\, d^3x' = P'^{\mu}(t'). \tag{2.21}$$

We now finally have all the ingredients we need to prove the theorem in Eq. 2.12.

Consider the '$\Leftarrow$'-implication first. From $\partial_{\nu} T^{\mu\nu} = 0$ everywhere, it follows, with the help of the theorem in Eq. 2.14, that T^{μ}_{Σ} is independent of Σ. In combination with Eq. 2.21, this implies that P^{μ} is conserved and transforms as a four-vector under Lorentz transformation. The argument goes as follows.

'P^{μ} is conserved' means that, in an arbitrary Lorentz frame with coordinates x^{μ} and for arbitrary times t_1 and t_2, $P^{\mu}(t_1) = P^{\mu}(t_2)$. Consider two hyperplanes $\Sigma(e^{\mu}, t_1)$ and $\Sigma(e^{\mu}, t_2)$ which are the hyperplanes of simultaneity in the x^{μ}-frame at t_1 and t_2, respectively. Using the definition of P^{μ} in Eq. 2.21, we can write

$$P^{\mu}(t_1) = T^{\mu}_{\Sigma(e^{\mu},t_1)} = T^{\mu}_{\Sigma(e^{\mu},t_2)} = P^{\mu}(t_2) \tag{2.22}$$

where I used that T^{μ}_{Σ} is independent of Σ. So, P^{μ} is indeed conserved.

'P^{μ} transforms as a four-vector under Lorentz transformation' means that for two arbitrary Lorentz frames, say, an x^{μ}-frame and an x'^{μ}-frame, whose coordinates are related via $x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$, $P'^{\mu} = \Lambda^{\mu}{}_{\nu}\, P^{\nu}$. Consider two hyperplanes Σ_1 and Σ_2, the former being a hyperplane of simultaneity in the x^{μ}-frame, the latter being a hyperplane of simultaneity in the x'^{μ}-frame. We can then write:

$$P'^{\mu} = T'^{\mu}_{\Sigma_2} = \Lambda^{\mu}{}_{\nu}\, T^{\nu}_{\Sigma_2} = \Lambda^{\mu}{}_{\nu}\, T^{\nu}_{\Sigma_1} = \Lambda^{\mu}{}_{\nu}\, P^{\nu}, \tag{2.23}$$

where I used Eq. 2.21 plus the fact that P^{μ} is time independent in the first and the last step and the fact that T^{μ}_{Σ} transforms as a four-vector and is independent of our choice of Σ in the steps in between. So, P^{μ} indeed transforms as a four-vector under Lorentz transformation.

This proves the 'if'-part of the theorem. The proof of the 'only if'-part, the '$\Rightarrow$'-implication in Eq. 2.12, is similar. Suppose P^{μ} is conserved and transforms as a four-vector under Lorentz transformation. It then follows (see above) that $T^{\mu}_{\Sigma_1} = T^{\mu}_{\Sigma_2}$ for arbitrary hyperplanes Σ_1 and Σ_2. From Eq. 2.18, it then follows that the integral of $\partial_{\nu} T^{\mu\nu}$ over the space-time volume enclosed between Σ_1 and Σ_2 vanishes. Since this is true for arbitrary Σ_1 and Σ_2, it has to be the case that the integrand vanishes everywhere, i.e., that $\partial_\nu T^{\mu\nu} = 0$ everywhere.

This concludes the proof of the theorem in Eq. 2.12. In Laue's terminology, the energy and momentum of a system are conserved and transform as a four-vector under Lorentz transformation if and only if the system is complete.

2.1.4 Laue's proof of the claim that the energy and momentum of a complete static system transform as a four-vector. I want to take a closer look at how Laue tried to prove the weaker result that the energy and momentum of a complete *static* system transform as a four-vector (the converse, as we have just seen, does not hold, and Laue does not claim it does). Consider (cf. Fig. 2.2) a system, not necessarily static, which is at rest[23] in a frame with coordinates x'^μ and moving at a velocity v in the x-direction with respect to a frame with coordinates x^μ, related to x'^μ via $x^\mu = \Lambda^\mu{}_\nu\, x'^\nu$, with

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.24}$$

Combine the energy and momentum of the system in the two frames in the quantities

$$P^\mu \equiv (U,\, c\mathbf{P}) \equiv \int T^{\mu 0} d^3x\ , \qquad P'^\mu \equiv (U',\, c\mathbf{P}') \equiv \int T'^{\mu 0} d^3x'\ , \tag{2.25}$$

where $T^{\mu\nu}$ is the sum of the energy-momentum tensors for the various constituents of the system under consideration. Laue now wants to show that for a complete static system

$$P^\mu = \Lambda^\mu{}_\nu\, P'^\nu. \tag{2.26}$$

He starts by writing the unprimed quantities in the expression for P^μ in terms of primed quantities. For the energy-momentum tensor he uses $T^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma T'^{\rho\sigma}$. For the volume element, he substitutes $d^3x = (1/\gamma)\, d^3x'$ (Laue 1911a, p. 144, the three equations above Eq. (8a)). This gives

$$P^\mu = \Lambda^\mu{}_\rho\, \Lambda^0{}_\sigma \frac{1}{\gamma} \int T'^{\rho\sigma} d^3x'. \tag{2.27}$$

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[23] Strictly speaking, we would have to say something like 'the system's center of mass is at rest in the x'^μ-frame' if we are talking about a non-static system.

Laue now investigates under what conditions Eq. 2.27 reduces to

$$P^{\mu} = \Lambda^{\mu}{}_{\nu} \int T'^{\nu 0} d^3x' = \Lambda^{\mu}{}_{\nu} P'^{\nu}. \tag{2.28}$$

In substituting $(1/\gamma)\, d^3x'$ for d^3x, Laue tacitly assumes that for *arbitrary* $T^{\mu\nu}$ one can replace the space integral in the x^{μ}-frame by a space integral in the x'^{μ}-frame. The space integral giving P^{μ} in the x^{μ}-frame, an integral over a hyperplane of simultaneity in the x^{μ}-frame, can, of course, be evaluated in the x'^{μ}-frame. However, one has to keep in mind that the hyperplane one is integrating over will not be a hyperplane of simultaneity in the x'^{μ}-frame, and hence not an ordinary space integral. If the system is static, one can prove, as I will do shortly, that the integral over the hyperplane of simultaneity in the x^{μ}-frame is equal to $1/\gamma$ times the integral over the hyperplane of simultaneity in the x'^{μ}-frame, justifying Laue's substitution of $(1/\gamma)\, d^3x'$ for d^3x, but in general this is certainly not the case.[24] The conclusion then is twofold. First, Laue's proof, as it stands, is in error (i.e., Eq. 2.27 does not follow from Eq. 2.25 for arbitrary $T^{\mu\nu}$, contrary to Laue's supposition). Second, it can easily be fixed. Laue invokes the assumption that the system be static in the next step of his proof (from Eq. 2.27 to Eq. 2.28). This retroactively justifies the move from Eq. 2.25 to Eq. 2.27.

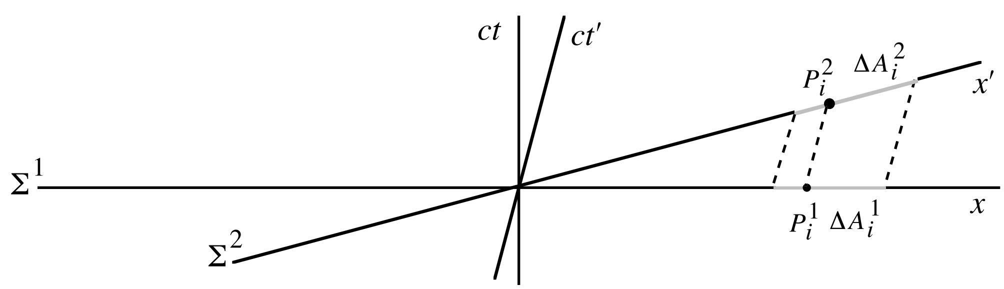

Figure 2.3 Relation between integrals over Σ^1 and Σ^2 for static systems.

The result needed to fill the gap in Laue's argument is not hard to prove. Consider Fig. 2.3. Let Σ^1 and Σ^2 be hyperplanes of simultaneity in the x^{μ}-frame and the x'^{μ}-frame, respectively. Divide Σ^1 and Σ^2 into corresponding segments (such as the shaded segments in the figure) connected by curves x' = constant, and with hypersurface areas ΔA_i^1 and ΔA_i^2. A simple

[24] Pauli (1921, pp. 126–127) compares Laue's derivation of the transformation of the four-momentum of static systems with an alternative derivation by Einstein (1907b). He does not comment on Laue's substitution of $(1/\gamma)\, d^3x'$ for d^3x, even though he emphasizes that Einstein in his derivation had to be careful keeping track of which hyperplane he was integrating over. It is clear that Einstein in 1907 had a much better understanding of these matters than Laue in 1911. In section 2.4, I will get back to this paper by Einstein.

argument[25] shows that $\Delta A_i^1 = \Delta A_i^2/\gamma$. Let P_i^1 and P_i^2 be corresponding space-time points in the i^{th} segments of Σ^1 and Σ^2, respectively.

Consider a tensor field $T^{\mu\nu\cdots\kappa\lambda}$ which is time independent in its rest frame:

$$\partial'_0 T'^{\mu\nu\cdots\kappa\lambda} = 0. \tag{2.29}$$

It follows that for arbitrary pairs (P_i^1, P_i^2) and in arbitrary x^μ-frames:

$$T^{\mu\nu\cdots\kappa\lambda}(P_i^1) = T^{\mu\nu\cdots\kappa\lambda}(P_i^2). \tag{2.30}$$

This, in turn, implies that

$$\int_{\Sigma^1} T^{\mu\nu\cdots\kappa\lambda}\, d\Sigma = \frac{1}{\gamma}\int_{\Sigma^2} T^{\mu\nu\cdots\kappa\lambda}\, d\Sigma, \tag{2.31}$$

as can be seen as follows. Write the integral on the left hand side as

$$\int_{\Sigma^1} T^{\mu\nu\cdots\kappa\lambda}\, d\Sigma = \lim_{\Delta A_i^1 \to 0} \sum_{i=1}^{\infty} T^{\mu\nu\cdots\kappa\lambda}(P_i^1)\, \Delta A_i^1. \tag{2.32}$$

Inserting $\Delta A_i^1 = \Delta A_i^2/\gamma$ and Eq. 2.30, we can rewrite the right hand side as

$$\frac{1}{\gamma} \lim_{\Delta A_i^2 \to 0} \sum_{i=1}^{\infty} T^{\mu\nu\cdots\kappa\lambda}(P_i^2)\, \Delta A_i^2 = \frac{1}{\gamma}\int_{\Sigma^2} T^{\mu\nu\cdots\kappa\lambda}\, d\Sigma, \tag{2.33}$$

which is just the right hand side of Eq. 2.31.

As a special case, consider the energy-momentum tensor $T^{\mu\nu}$ for a static system, i.e., $\partial'_0 T'^{\mu\nu} = 0$. From the general result in Eq. 2.31, it then follows that:

$$\begin{aligned} P^\mu &\equiv \int T^{\mu 0} d^3x \\ &= \int_{\Sigma^1} T^{\mu 0} d\Sigma \\ &= \frac{1}{\gamma}\int_{\Sigma^2} T^{\mu 0} d\Sigma \end{aligned} \tag{2.34}$$

---

[25] See the discussion following Eq. 2.78 below.

$$= \frac{1}{\gamma} \Lambda^{\mu}{}_{\rho} \Lambda^{0}{}_{\sigma} \int T'^{\rho\sigma} d^3x',$$

which is just Eq. 2.27, the problematic step in Laue's proof of his claim that the energy and momentum of a complete static system transform as a four-vector.

It is instructive to consider Laue's argument a little more closely. Not only will this show us why Laue himself felt he needed the condition that the system be static as well as complete, we will also see that Laue derived a very interesting result for complete static systems in the course of his argument, a result known as "Laue's theorem."[26] Consider the 0-component of Eq. 2.27 (= Eq. 2.34). Inserting Eq. 2.24 for $\Lambda^{\mu}{}_{\nu}$, we find

$$\begin{aligned} P^0 &= \frac{1}{\gamma} \Lambda^{0}{}_{\rho} \Lambda^{0}{}_{\sigma} \int T'^{\rho\sigma} d^3x' \\ &= \Lambda^{0}{}_{\rho} \int T'^{\rho 0} d^3x' + \gamma\beta \int T'^{01} d^3x' + \gamma\beta^2 \int T'^{11} d^3x'. \end{aligned} \tag{2.35}$$

The first term in the expression on the second line is equal to $\Lambda^{0}{}_{\nu} P'^{\nu}$, the 0-component of the right hand side of Eq. 2.26, so Laue needs to introduce some extra conditions to make sure the remaining terms vanish. For static systems, the 0*i*-components of $T^{\mu\nu}$ vanish in the system's rest frame. That takes care of the second term. For a *complete* static system, one has

$$\partial'_i T'^{ij} = - \partial'_0 T'^{0j} = 0. \tag{2.36}$$

From Gauss's theorem, it then follows that the surface integral

$$\int_S T'^{ij} n'_i \, dS , \tag{2.37}$$

where the n'^i-s are the components of the outward pointing normal vector on the surface S in the x'^{μ}-frame, vanishes for arbitrary closed surfaces. By cleverly choosing particular closed surfaces, Laue shows that the volume integrals

$$\int T'^{ij} d^3x' \tag{2.38}$$

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[26] See Norton 1992a, p. 52; 1993a, pp. 16–17. Miller (1981, p. 373) writes that it was Mie who christened the result "Laue's theorem" in 1913.

all vanish (Laue 1911a, p. 152). So, for complete static systems, the last term in Eq. 2.35 also vanishes. This result is interesting in its own right: for a complete static system in its rest frame, the space integral over the stresses always vanishes. This is called “Laue’s theorem.”

2.2 The '4/3-puzzle' of the Lorentz-Poincaré electron and the Laue effect: the Laue definition of four-momentum versus the Rohrlich definition

2.2.1 The relation between the Trouton-Noble experiment and the Lorentz-Poincaré electron. Laue applied his claim that the energy and momentum of a complete static system transform as a four-vector to two special cases, the Trouton-Noble experiment and the electron model of Lorentz and Poincaré. The latter case has attracted far more attention than the former, both from physicists and historians, especially because of a notorious puzzle associated with it. The electron's electromagnetic rest mass times c^2 is not equal to its electromagnetic energy, but to $^4/_3$ times that energy. Laue's discussion of the Lorentz-Poincaré electron as a complete static system strongly contributed to what would remain the standard solution of this '$^4/_3$-puzzle' until the work of Rohrlich (1960, 1965).

The analysis of the transformation of integrals over spacelike hyperplanes given above provides us with all the tools we need to develop the Rohrlich solution of the puzzle. I will then apply the same ideas in the context of the Trouton-Noble experiment. Somewhat surprisingly perhaps, given the close connection between the two problems and the widespread acceptance of the Rohrlich solution of the $^4/_3$-puzzle, this has never been done before, at least not to my knowledge.[27]

2.2.2 A brief history of '4/3-puzzle' of the Lorentz-Poincaré electron.[28] We need to take a brief look at the tangled history of the $^4/_3$-puzzle. In Lorentz's original model (Lorentz 1904b[29]), the electron is conceived of as a purely electromagnetic entity. It consists of nothing but a static spherical charge distribution which is subject to the Lorentz-FitzGerald contraction and whose mass is due solely to the interaction with its self-field. Suppose the x^μ-frame is at rest in the ether and the electron is at rest with respect to the x'^μ-frame, which is itself moving with respect to the x^μ-frame at some constant velocity $\mathbf{v}$ in the positive x-direction (so, we have $x^\mu = \Lambda^\mu{}_\nu\, x'^\nu$, where $\Lambda^\mu{}_\nu$ is given by Eq. 2.24). Using Eq. 2.27 (which is justified in this case because the system is static) to compute (c times) the x-component of the electron's momentum in the x^μ-frame, we find[30]

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[27] A recent monograph (Yaghjian 1992) on what the author refers to as the "Lorentz-Abraham" model of the electron, with an approving preface by Rohrlich, does not even mention the Trouton-Noble experiment.

[28] See Miller 1981, pp. 73–86, for a more extensive discussion of the roles of Lorentz, Abraham, and Poincaré in the history of this issue.

[29] Yaghjian (1992, p. 1) is wrong when he dates the model back to 1892.

[30] In section 3.4, we will have occasion to go through this calculation, using Lorentz's theorem of corresponding states. See Eqs. 3.112–3.113 for $l = 1$.

$$P^1 \equiv \int T^{10}\, d^3x = \frac{1}{\gamma}\Lambda^1{}_\rho \Lambda^0{}_\sigma \int T'^{\rho\sigma}\, d^3x' = \frac{4}{3}\gamma \frac{U'}{c^2}\beta, \tag{2.39}$$

where U' is the electron's electromagnetic energy in the x'^μ-frame. This means that the four-momentum of Lorentz's electron does not transform as a four-vector, as the four-momentum of a relativistic particle should. If it did, we would have $P^1 = \Lambda^1{}_\mu P'^\mu$. With $P'^\mu = (U', 0, 0, 0)$ and $\Lambda^1{}_0 = \gamma\beta$, this would give

$$P^1 = \gamma \frac{U'}{c^2}\beta \tag{2.40}$$

which differs by a factor $^4\!/_3$ from Eq. 2.39. The problem is more vividly stated in terms of the electron's mass-energy relation. In the non-relativistic limit, cP^1, the x-component of momentum, should reduce to its Newtonian form $cP^1 = m'v$, where m' is the electron's rest mass. However, the non-relativistic limit of Eq. 2.39 gives $cP^1 = {}^4\!/_3(U'/c^2)v$. This gives ${}^4\!/_3U' = m'c^2$ instead of $U' = m'c^2$!

This puzzling factor $^4\!/_3$ was first discovered, albeit in a somewhat different guise, by Abraham (1904, 1905),[31] who used it to argue that the mass of Lorentz's electron cannot be of purely electromagnetic origin. For a staunch supporter of the electromagnetic view of nature, such as Abraham, this was a damning objection.

Lorentz and Poincaré accepted Abraham's diagnosis of the problem, but, unlike Abraham, were willing to give up the notion that the electron is a purely electromagnetic structure. Poincaré (1906) explicitly added a non-electromagnetic ether pressure term to the energy-momentum tensor for the system, the so-called Poincaré stresses. He plausibly argued that such a term was needed to counterbalance the Coulomb repulsion in the electron. Poincaré showed that this extra term gives a contribution of $-{}^1\!/_3\gamma(U'/c^2)\beta$ to P^1 in Eq. 2.39, thus reducing the puzzling factor $^4\!/_3$ to unity.

When the Poincaré stresses are included, Lorentz's electron behaves as a point mass in relativistic mechanics. On the basis of his claim that the energy and momentum of a complete static system transform as a four-vector, Laue (1911a, pp. 152–153[32]) argued that Lorentz's spherical charge distribution rendered stable by Poincaré's ether pressure is just one among infinitely many conceivable models with this property. Given that Laue's claim about complete

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[31] What Abraham discovered was that Lorentz's model of a purely electromagnetic electron leads to a contradiction. He showed that the electron's so-called longitudinal mass derived from its electromagnetic momentum differs by a factor $^4\!/_3$ from the longitudinal mass derived from its electromagnetic energy (see Miller 1981, pp. 75–79, and section 3.4).

[32] This passage is discussed in Norton 1992a, pp. 52–53.

static systems is indeed true, this vindicates the response of Einstein (1907a) to a query by Ehrenfest (1907) whether the energy and momentum of an electron would still transform as the energy and momentum of a relativistic point mass if it were non-spherical.[33] Laue wrote that as long as the electron is a complete static system it would. We have seen that the electron does not even have to be in static equilibrium. It only needs to be a complete or closed system. What I want to emphasize at this point is that Laue's discussion reinforced the impression, given by Poincaré, that the problem concerning the transformation properties of the electron's four-momentum is inseparably connected to the problem of its stability.

This view of the $^4/_3$-puzzle made its way into Pauli's influential review article (Pauli 1921, pp. 185–187[34]), and remained standard until the work of Rohrlich (1960, 1965). Rohrlich traced the factor $^4/_3$ to the standard definition of four-momentum that Laue and Pauli—and, in effect, ether theorists such as Abraham, Lorentz, and Poincaré—used. Rohrlich showed how the problem concerning the transformation of the electron's four-momentum could be solved independently of the stability problem by introducing an alternative definition of four-momentum. Rohrlich found that these results had already been discovered, forgotten, and rediscovered. He credits Fermi with the original discovery in 1922. Fermi's work went unnoticed as did that of two rediscoverers, Wilson in 1936, and Kwal in 1949 (Rohrlich 1965, p. 17). Rohrlich made sure the results would not be forgotten again.[35]

2.2.3 The Rohrlich and Laue definitions of the four-momentum of spatially extended systems. Rohrlich's solution of the $^4/_3$-puzzle rests on a careful analysis of the definition of the electron's four-momentum. There is a certain arbitrariness in the way we give a relativistic definition of such quantities as the four-momentum and angular momentum of spatially extended open systems, such as a static charge distribution and its field. To be sure, part of the definition is unproblematic. We define such quantities as integrals of corresponding densities (the energy-momentum tensor in the case of four-momentum) over spacelike hyperplanes. So,

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33 For discussions of this exchange between Einstein and Ehrenfest, see Klein 1967, pp. 515–516, McCormmach 1970b, pp. 488–489, Miller 1981, pp. 235–236, and Norton 1992a, p. 45.

34 Parenthetically, I may add that Pauli did mention the Trouton-Noble experiment in this context. After explaining Ehrenfest's query about non-spherically shaped electrons, he writes: "It was stressed by Laue that the situation is quite analogous to that in Trouton and Noble's experiment" (Pauli 1921, p. 186). In a footnote appended to this comment, he refers to Laue 1911a.

35 In the preface to the second edition of his textbook on special relativity, for instance, Aharoni (1965) cites Rohrlich (1960) as having motivated some major revisions in the first edition of his book, published in 1959. In this preface, Aharoni also adds another name to Rohrlich's list of rediscoverers: Dirac. This makes it the second most famous discovery, I guess, with which both Fermi and Dirac can be credited. For a very clear and concise exposition of Rohrlich's results, see Aharoni 1965, section 5.5, pp. 160–165.

in some arbitrary x^μ-frame, the four-momentum of a spatially extended system will be a quantity of the form (cf. Eq. 2.13)[36]

$$T^\mu_{\Sigma(n^\mu,\,\tau)} \equiv \int_{\Sigma(n^\mu,\,\tau)} T^{\mu\nu} n_\nu \, d\Sigma, \tag{2.41}$$

where $\Sigma(n^\mu, \tau)$ is some spacelike hyperplane. If $T^{\mu\nu}$ is the energy-momentum tensor of a closed system (i.e., $\partial_\nu T^{\mu\nu} = 0$), T^μ_Σ is independent of Σ (see the theorem in Eq. 2.14). This means that we can simply identify $P^\mu(t)$ with T^μ_Σ for any Σ we want. However, if $T^{\mu\nu}$ is the energy-momentum tensor of an open system (i.e., $\partial_\nu T^{\mu\nu} \neq 0$)—such as, say, the ether pressure part of the Lorentz-Poincaré electron or the electromagnetic part of the condenser system in the Trouton-Noble experiment—then the theorem in Eq. 2.14 tells us that T^μ_Σ will depend on our choice of Σ. Unless we are happy with a definition that makes the four-momentum of spatially extended open systems a hyperplane dependent quantity,[37] we need a convention that tells us for which hyperplane Σ the quantity T^μ_Σ in Eq. 2.41 is to be identified with the four-momentum $P^\mu(t)$ in the x^μ-frame. I will consider two such conventions (cf. Fig. 2.4).

We already encountered the standard convention. In the standard definition of $P^\mu(t)$ (see Eq. 2.10), $\Sigma(n^\mu, \tau)$ is chosen to be the hyperplane of simultaneity at time t in the x^μ-frame we happen to be using. For the normal n^μ one picks the vector with components (1, 0, 0, 0) in the x^μ-frame, for τ one simply picks t. Introducing the notation $\Sigma^{\text{sim}}(x^\mu)$ for such hyperplanes, we can write the standard definition as[38]

$$P^\mu_{\text{L}}(t) \equiv T^\mu_{\Sigma(n^\mu,\,\tau)\,=\,\Sigma^{\text{sim}}(x^\mu)}, \tag{2.42}$$

where the subscript 'L' refers to Laue. By this I do not mean to imply that Laue was the first (nor the last, obviously) to use this definition. It can be traced back to the work of such ether

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[36] Cf. Rohrlich 1960, p. 641, Eq. (11); 1965, p. 89, Eq. (4-123)

[37] Gordon Fleming has shown that embracing such hyperplane dependence solves a number of notorious puzzles in relativistic quantum theory. It explains, for instance, the curious transformation properties of the so-called Newton-Wigner position operator in its standard non-hyperplane-dependent definition. Fleming has argued that a Lorentz invariant theory of state vector reduction calls for a definition of state vectors that would make state vectors hyperplane dependent quantities (see, e.g., Fleming 1989, 1994). For a critical response, see Maudlin 1994. I will not pursue the approach suggested by Fleming's work here. For my main purpose in this context (which is to establish the kinematical nature of the Laue effect in special relativity), working with Rohrlich's convention for picking hyperplanes, which I will introduce shortly, is as good as working with general hyperplane dependence. If, however, one's goal is to eradicate the arbitrariness from the definition of all quantities involving integration over hyperplanes, Fleming's approach, it seems to me, is a very natural and elegant way to proceed.

[38] Cf. Rohrlich 1960, p. 640, Eq. (5); 1965, p. 132, Eqs. (6-33)–(6-34); Aharoni 1965, p. 163, Eq. (5.19).

theorists as Lorentz and Abraham.[39] What speaks for this particular definition is that it does seem to capture what we mean by 'the four-momentum of the system in the x^μ-frame.' What speaks against it is that it is not Lorentz invariant. Compare P^μ_L in the x^μ-frame and P'^μ_L in the x'^μ-frame. The notation suggests that P^μ_L and P'^μ_L represent the *same* quantity in different coordinates. This is not the case. P^μ_L and P'^μ_L are integrals over different hyperplanes. They represent *different* quantities in different coordinates. It should not be surprising, therefore, that $P^\mu_L \neq \Lambda^\mu{}_\nu P'^\nu_L$ for the electromagnetic part of the four-momentum of the Lorentz-Poincaré electron (see Eqs. 2.39–2.40). What is more surprising, in fact, is that P^μ_L only differs from $\Lambda^\mu{}_\nu P'^\nu_L$ by a factor of $4/3$, at least as far as the spatial components are concerned.[40] This is because of the spherical symmetry of the situation. In the case of the condenser in the Trouton-Noble experiment, as we will see in section 2.3, the four-momentum P^μ_L of, say, the electromagnetic part of the system will not even have the same direction as $\Lambda^\mu{}_\nu P'^\nu_L$.

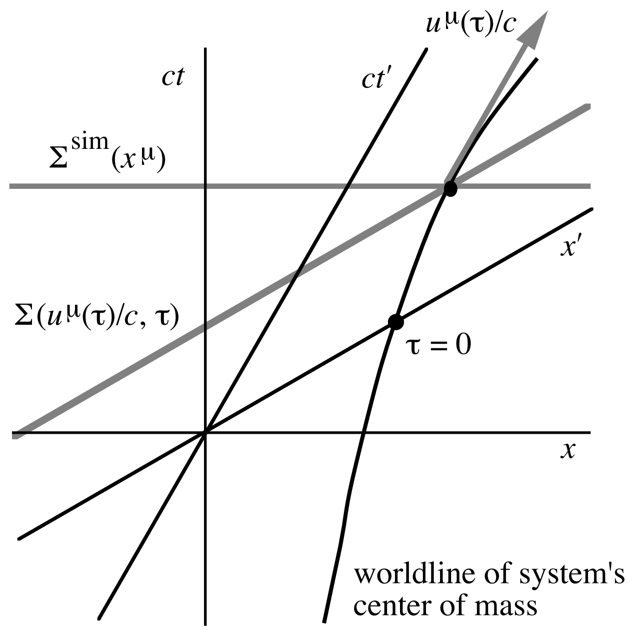

Figure 2.4 Two different conventions for choosing spacelike hyperplanes in the definition of four-momentum $P^\mu(t)$ for spatially extended systems in an arbitrary x^μ-frame.

Rohrlich proposes a different convention for picking the relevant hyperplanes. The resulting definition of $P^\mu(t)$ turns out to be particularly convenient for static systems, such as those in the cases we are interested in (the Lorentz-Poincaré electron and the Trouton-Noble experiment).

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[39] In fact, Rohrlich (1960, p. 640) and Aharoni (1965, p. 163) refer to the definition in Eq. 2.42 (written in a form that is much closer to how it would have been written at the time) as the 'Abraham-Lorentz'-definition.

[40] The relation between the 0-components of P^μ_L and $\Lambda^\mu{}_\nu P'^\nu_L$ is a little more complicated: $P^0_L = \gamma U' (1 + \beta^2/3)$ whereas $\Lambda^0{}_\nu P'^\nu_L = \gamma U'$ (see section 3.4).

Rohrlich only formulated his convention for such systems. I will formulate it for arbitrary systems, mainly to show that it is a genuine alternative to the standard convention in all cases. The basic idea of Rohrlich's proposal is to choose a hyperplane of simultaneity in the system's rest frame no matter in which x^μ-frame we want to define the four-momentum $P^\mu(t)$. For static systems this means that we pick hyperplanes $\Sigma(n^\mu, \tau)$, where for n^μ we pick the system's (constant) four-velocity u^μ divided by c, and for τ we pick some arbitrary value.[41] This definition can easily be generalized to non-static situations (cf. Fig. 2.4). Consider the worldline of the center of mass of some arbitrary system.[42] Pick some arbitrary zero point for the proper time measured along that worldline. That is all we need to specify our alternative convention about picking the hyperplane $\Sigma(n^\mu, \tau)$ in the definition of P^μ at some arbitrary time t in some arbitrary x^μ-frame. For τ we pick the proper time associated with the point on this worldline that has x^0-coordinate ct. For n^μ we pick $u^\mu(\tau)$, the instantaneous four-velocity at that point divided by c. With these stipulations, our alternative definition of four-momentum can be written as

$$P^\mu_{\mathrm{R}}(t) \equiv T^\mu_{\Sigma(n^\mu,\,\tau)\,=\,\Sigma(u^\mu(\tau)/c,\,\tau)}, \tag{2.43}$$

where the subscript 'R' refers to Rohrlich. The virtues and shortcomings of this definition are just the reverse of those of the definition in Eq. 2.42. On the positive side, under Rohrlich's definition P^μ transforms as a four-vector no matter whether we are dealing with a closed or an open system. $P^\mu_{\mathrm{R}}(t)$ in the x^μ-frame and $P'^\mu_{\mathrm{R}}(t')$ in the x'^μ-frame represent the same quantity in different coordinates. In other words, the Rohrlich definition, unlike the definition used by Laue, is Lorentz invariant. Even for open systems, such as the electromagnetic part of the Lorentz-Poincaré electron, it gives $P^\mu_{\mathrm{R}} = \Lambda^\mu{}_\nu P'^\nu_{\mathrm{R}}$. There is no mysterious factor $4/3$. On the negative side, the physical interpretation of $P^\mu_{\mathrm{R}}(t)$ is problematic. Unless the x^μ-frame happens to be the system's rest frame, $P^\mu_{\mathrm{R}}(t)$ will not be an integral over a hyperplane of simultaneity in that frame. Hence, it will not be an ordinary space integral. Yet, this seems to be what we mean by 'the four-momentum of the system in the x^μ-frame.'

Fortunately, Rohrlich's solution of the $4/3$-puzzle does not depend on the alternative definition of four-momentum in Eq. 2.43 being superior to the standard definition in Eq. 2.42. If it did, we would have a problem, for, as we have seen, both definitions have pros and cons. All that is needed for the solution of the puzzle is the recognition that in the final analysis it is a matter of convention whether we use one definition or the other. Under the standard definition,

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[41] Cf. Rohrlich 1960, p. 641, Eqs. (13) and (16); 1965, p. 130, Eqs. (6-18) and (6-19); Aharoni 1965, p. 161, Eq. (5.6).

[42] As Gordon Fleming pointed out to me, we need to be careful about how we evaluate the center of mass. All that matters for my purposes is that we do so in a way that the center of mass can be said to be at rest in the system's instantaneous rest frame.

we find the $^4/_3$-problem. Under the alternative definition, we do not. The problem of the electron's stability is present under both definitions. These two problems thus have a very different status. The latter is a genuine problem (we do need some non-electromagnetic structure to prevent the electron's charge distribution from flying apart), the former is nothing but an artifact of a particular way of defining four-momentum. As Rohrlich sums up the case: "There is no relation between the need for cohesive forces and the factor $^4/_3$. We seem to have a relativistic but unstable electron" (Rohrlich 1965, p. 17).[43]

2.2.4 Applying Rohrlich's insights to the Trouton-Noble experiment. The argument I just spelled out for the case of the Lorentz-Poincaré electron can easily be adapted to the case of the Trouton-Noble experiment. The condenser in the Trouton-Noble experiment, like the Lorentz-Poincaré electron, consists of a static charge distribution, the electromagnetic field generated by that charge distribution, and some non-electromagnetic structure preventing the charges from moving. The difference is that in the case of the Trouton-Noble experiment we do not have the simple spherical symmetry we have in the case of the electron. As a consequence, there will be more striking differences between the picture of what happens in the Trouton-Noble condenser based on the standard definition of four-momentum in Eq. 2.42 that Laue used and the picture based on Rohrlich's alternative definition in Eq. 2.43. In particular, we will find that the Laue effect—the delicately balanced turning couples coming from the various constituents of a complete static system—is found only in the Laue picture. In the Rohrlich picture, we find no such thing. So, the Laue effect has the same status as the factor $^4/_3$ in the case of the electron. It is an artifact of a particular way of defining four-momentum.

Since the Laue effect involves angular momentum rather than four-momentum, the argument that the effect is an artifact of a convention about how to choose spacelike hyperplanes is best made in terms of angular momentum. The relativistic definition of the angular momentum of spatially extended open systems involves the same conventional element we encountered in the relativistic definition of the four-momentum of such systems.

2.2.5 The Rohrlich and Laue definitions of the angular momentum of spatially extended systems. Relativistically, the angular momentum of some spatially extended system

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[43] Let me give a concise summary of the situation. The electromagnetic part of the Lorentz-Poincaré electron is an open system, i.e., $\partial_\nu T^{\mu\nu}_{\rm EM} \neq 0$. It follows that (a) a purely electromagnetic electron would be unstable, and (b) the electromagnetic four-momentum, defined as an ordinary space integral over $T^{\mu 0}_{\rm EM}$, does not transform as a four-vector. Solving problem (a) by adding a non-electromagnetic part to close the system, i.e., adding a term $T^{\mu\nu}_{\text{non-EM}}$ such that $\partial_\nu (T^{\mu\nu}_{\rm EM} + T^{\mu\nu}_{\text{non-EM}}) = 0$, automatically takes care of problem (b). Therefore, it is easy to lose sight of the fact that (a) and (b) are separate problems, and that problem (b) can actually be defined away, showing that, unlike problem (a), it was not much of a problem to begin with.

in some arbitrary x^μ-frame will be represented by a quantity of the form (cf. Eq. 2.13 and Eq. 2.41)[44]

$$J^{\mu\nu}_{\Sigma} \equiv \int_{\Sigma} J^{\mu\nu\lambda}\, n_\lambda \, d\Sigma\,, \tag{2.44}$$

where Σ is an yet unspecified spacelike hyperplane and where $J^{\mu\nu\lambda}$ is the so-called angular momentum current defined as[45]

$$J^{\mu\nu\lambda} \equiv x^{\mu} T^{\nu\lambda} - x^{\nu} T^{\mu\lambda} \tag{2.45}$$

Under very general conditions, the canonical and the symmetric energy-momentum tensors will differ only by a term that is conserved separately. In that case, we are free to use the symmetric energy-momentum tensor.[46] With such a symmetric energy-momentum tensor ($T^{\mu\nu} = T^{\nu\mu}$) for a complete system ($\partial_\nu T^{\mu\nu} = 0$), the angular momentum current is divergence free:

$$\partial_\lambda J^{\mu\nu\lambda} = x^{\mu}\partial_\lambda T^{\nu\lambda} + T^{\nu\mu} - x^{\nu}\partial_\lambda T^{\mu\lambda} - T^{\mu\nu} = 0 \tag{2.46}$$

According to the general theorem in Eq. 2.14, $\partial_\lambda J^{\mu\nu\lambda} = 0$ if and only if $J^{\mu\nu}_{\Sigma}$ is independent of Σ. So, for a closed system with a symmetric energy-momentum tensor, we can define angular momentum simply as

$$J^{\mu\nu}(t) \equiv J^{\mu\nu}_{\Sigma}, \tag{2.47}$$

for arbitrary Σ. It immediately follows from this Σ-independence that the total angular momentum of a closed system with a symmetric total energy-momentum tensor is conserved (cf. Eq. 2.22). Hence, there will be no net turning couple on such systems. *A fortiori*, there will be no net turning couple on complete *static* systems with a symmetric total energy-momentum tensor, such as the system in the Trouton-Noble experiment.

Whether or not the constituents of a complete static system give rise to (delicately balanced) turning couples depends on our definition of the angular momentum for open systems. For open systems, the theorem in Eq. 2.14 tells us, $J^{\mu\nu}_{\Sigma}$ depends on Σ. So, we need a convention

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[44] Cf. Rohrlich 1965, p. 95, Eq. (4-155)–(4-156).

[45] A very thorough discussion of how to define the angular momentum current as the Noether current (and the angular momentum tensor as the corresponding Noether charge) associated with Lorentz invariance of the Lagrangian can be found in Soper 1976, pp. 101-123. The canonical angular momentum current constructed out of the canonical energy-momentum tensor typically has a "spin" part in addition to an "orbital angular momentum" part of the form given in Eq. 2.25. However, there is no "spin" part when we use the symmetric energy-momentum tensor to construct the angular momentum current as we did in Eq. 2.25 (ibid., pp. 118-120).

[46] I am grateful to Tony Duncan for clarifying this point.

about picking a hyperplane Σ if we want to give a definition of angular momentum that applies both to open and to closed systems. As with four-momentum, I will consider two such conventions (cf. Fig. 2.4),[47] leading to two different definitions of angular momentum. In what I will call the 'Laue definition,' one picks hyperplanes with *different* orientations n^μ in different x^μ-frames (viz., hyperplanes of simultaneity in the x^μ-frame under consideration; cf. the definition of four-momentum in Eq. 2.42):

$$J_{\mathrm{L}}^{\mu\nu}(t) \equiv J^{\mu\nu}_{\Sigma(n^\mu,\tau) = \Sigma^{\mathrm{sim}}(x^\mu)}. \tag{2.48}$$

In what I will call the 'Rohrlich definition' one picks hyperplanes with the *same* orientation n^μ in different x^μ-frames (viz., hyperplanes of simultaneity in the system's rest frame; cf. the definition of four-momentum in Eq. 2.43):

$$J_{\mathrm{R}}^{\mu\nu}(t) \equiv J^{\mu\nu}_{\Sigma(n^\mu,\tau) = \Sigma(u^\mu(\tau)/c,\tau)}. \tag{2.49}$$

The turning couple $\tau^{\mu\nu}$ a system exerts on itself is minus the turning couple we would have to exert to change the system's angular momentum (cf. section 1.4, Eq. 1.37). Hence,

$$\tau^{\mu\nu} \equiv -\frac{dJ^{\mu\nu}}{dt} \tag{2.50}$$

The definition of the turning couple $\tau^{\mu\nu}$ obviously inherits the convention about choosing spacelike hyperplanes from the definition of angular momentum $J^{\mu\nu}$, so we need to distinguish between $\tau_{\mathrm{L}}^{\mu\nu}$ and $\tau_{\mathrm{R}}^{\mu\nu}$. Under both definitions of $J^{\mu\nu}$ in Eqs. 2.48–2.49, the total angular momentum of a complete system is conserved and the net turning couple vanishes:

$$\tau^{\mu\nu}_{\mathrm{tot_L}} = \tau^{\mu\nu}_{\mathrm{tot_R}} = 0 \tag{2.51}$$

Consider a complete static system consisting of an electromagnetic and a non-electromagnetic part, both of which are open sub-systems. In the Trouton-Noble experiment we have an example of this situation. What can we say about the turning couples $\tau^{\mu\nu}_{\mathrm{EM}}$ and $\tau^{\mu\nu}_{\mathrm{non\text{-}EM}}$ coming from the two parts of the system, considered separately? I will show that under the Laue definition we find what I called the Laue effect, i.e.,

$$\tau^{\mu\nu}_{\mathrm{EM_L}} = -\tau^{\mu\nu}_{\mathrm{non\text{-}EM_L}} \neq 0, \tag{2.52}$$

whereas under the Rohrlich definition we do not, i.e.,

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[47] As with four-momentum, I will not pursue the option of representing angular momentum by a hyperplane dependent quantity.

$$\tau^{\mu\nu}_{\mathrm{EM_R}} = \tau^{\mu\nu}_{\mathrm{non\text{-}EM_R}} = 0. \tag{2.53}$$

To prove the claim in Eq. 2.53, I will show that, under the Rohrlich definition, the angular momentum of a static system, open or closed, is always conserved. But first, I will prove the claim in Eq. 2.52 by showing that $J^{\mu\nu}_{\mathrm{EM_L}}$ and $\tau^{\mu\nu}_{\mathrm{EM_L}}$ (or rather their *ij*-components) are just alternative ways of writing the electromagnetic angular momentum **L** and the turning couple **T** we encountered in the discussion of Lorentz's account of the Trouton-Noble experiment in section 1.4.

Notice that $J^{\mu\nu}_{\mathrm{L}}$ can be written as an ordinary space integral in every x^μ-frame:

$$J^{\mu\nu}_{\mathrm{L}}(t) = \int_{\Sigma^{\mathrm{sim}}(x^\mu)} J^{\mu\nu\lambda}(x^\alpha)\, n_\lambda\, d\Sigma = \int J^{\mu\nu 0}(t,\, \mathbf{x})\, d^3x \tag{2.54}$$

Inserting Eq. 2.45 for the angular momentum current $J^{\mu\nu\lambda}$ into ($1/c$ times) the *ij*-components of $J^{\mu\nu}_{\mathrm{L}}$, we find:

$$\frac{1}{c} J^{ij}_{\mathrm{L}}(t) = \frac{1}{c}\int x^i\, T^{j0}(t,\, \mathbf{x}) - x^j\, T^{i0}(t,\, \mathbf{x})\, d^3x\ . \tag{2.55}$$

It is easily verified that this expression is just an alternative way of writing the classical angular momentum associated with the momentum density $p^i \equiv T^{i0}/c$. The integrand in Eq. 2.55 can be rewritten as

$$x^i p^j - x^j p^i = \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right) x^l p^m = \varepsilon^{kij}\left(\varepsilon_{klm} x^l p^m\right). \tag{2.56}$$

The factor in parentheses in the last term in Eq. 2.56 is the *k*-component of the classical angular momentum density $\mathbf{l} = \mathbf{x} \times \mathbf{p}$. So, we can rewrite Eq. 2.55 as

$$\frac{1}{c} J^{ij}_{\mathrm{L}}(t) = \int \varepsilon^{ijk}\left(\mathbf{x} \times \mathbf{p}(t,\, \mathbf{x})\right)_k d^3x \ = \varepsilon^{ijk} L_k(t), \tag{2.57}$$

where the angular momentum **L** is defined simply as the space integral over the corresponding density **l**. The *ij*-components of the turning couple $\tau^{\mu\nu}_{\mathrm{L}}$ can likewise be rewritten in terms of the turning couple **T** we considered in chapter one:

$$\frac{1}{c}\tau^{ij}_{\mathrm{L}} = -\frac{1}{c}\frac{dJ^{ij}_{\mathrm{L}}}{dt} = -\,\varepsilon^{ijk}\frac{dL_k}{dt} = \varepsilon^{ijk}\, T_k. \tag{2.58}$$

With the help of Eqs. 2.57–2.58, the *ij*-components of the equation for the turning couple $\tau_{\mathrm{L}}^{\mu\nu}$ can be rewritten as Eq. 1.37 which we used in section 1.4 as a short-cut for Lorentz's derivation of the turning couple coming from the electromagnetic forces:

$$\frac{1}{c}\,\tau_{\mathrm{L}}^{ij} = -\frac{1}{c}\,\frac{dJ_{\mathrm{L}}^{ij}}{dt} \;\Leftrightarrow\; \varepsilon^{ijk}\,T_k = -\,\varepsilon^{ijk}\,\frac{dL_k}{dt} \;\Leftrightarrow\; \mathbf{T} = -\,\frac{d\mathbf{L}}{dt} \tag{2.59}$$

For the electromagnetic turning couple, we thus recover the result found by Lorentz (cf. Eqs. 1.58–Eq. 1.62)

$$\mathbf{T}_{\mathrm{EM}} = -\,\frac{d\mathbf{L}_{\mathrm{EM}}}{dt} = -\frac{d}{dt}\int \mathbf{x}\times\mathbf{g}(t,\,\mathbf{x})\,\mathrm{d}^3\mathrm{x} = -\,\mathbf{v}\,\times\mathbf{G}, \tag{2.60}$$

where **g** is the electromagnetic momentum density and **G** is the electromagnetic momentum, the spatial components of the electromagnetic four-momentum of the system under the Laue definition in Eq. 2.42. As we saw in section 1.4, $\mathbf{T}_{\mathrm{EM}} \neq 0$ (see Eq. 1.39). This proves the claim in Eq. 2.52 that $\tau_{\mathrm{EM_L}}^{\mu\nu} \neq 0$. Using Eq. 2.58, we can write

$$\tau_{\mathrm{EM_L}}^{ij} = c\;\varepsilon^{ijk}\,T_k^{\mathrm{EM}} \neq 0 \tag{2.61}$$

Since $\tau_{\mathrm{tot_L}}^{ij} = 0$ (see Eq. 2.51), it follows that $\tau_{\text{non-EM}_\mathrm{L}}^{\mu\nu} = -\,\tau_{\mathrm{EM_L}}^{\mu\nu} \neq 0$.

Under the Rohrlich definition, there will be no such turning couples. The time derivative of $J_{\mathrm{R}}^{\mu\nu}(t)$ giving the turning couple $\tau_{\mathrm{R}}^{\mu\nu}$ will be zero for static systems, open as well as closed. The crucial input for proving this result is that $J_{\mathrm{R}}^{\mu\nu}(t)$ always transforms as a tensor, no matter whether a system is open or closed, static or non-static. Let the x'^{μ}-frame be the rest frame of some static system, related to some arbitrary x^{μ}-frame in which we happen to evaluate $J_{\mathrm{R}}^{\mu\nu}(t)$ through $x^{\mu} = \Lambda^{\mu}{}_{\nu}\, x'^{\nu}$. Since $J_{\mathrm{R}}^{\mu\nu}(t)$ transforms as a tensor, we have

$$J_{\mathrm{R}}^{\mu\nu}(t) = \Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma}\,J'^{\rho\sigma}_{\mathrm{R}}(t') \tag{2.62}$$

So, $J_{\mathrm{R}}^{\mu\nu}(t)$ is independent of t if and only if $J'^{\mu\nu}_{\mathrm{R}}(t')$ is independent of t'. One easily verifies that $J'^{\mu\nu}_{\mathrm{R}}(t')$ is indeed independent of t' for a static system. In the system's rest frame, the definition of angular momentum in Eq. 2.49 takes on a particularly simple form because $u'^{\mu}/c = (1,\,0,\,0,\,0)$ and τ can be set equal to t' (cf. Fig. 2.4). This gives

$$
\begin{aligned}
J'^{\mu\nu}_{\mathrm{R}}(t') &\equiv J'^{\mu\nu}_{\Sigma\left(n'^{\mu},\,\tau\right) = \Sigma\left(u'^{\mu}(\tau)/c\,=\,(1,\,0,\,0,\,0)\;,\,\tau\,=\,t'\right)} \\
&= \int_{\Sigma} J'^{\mu\nu\lambda}(x'^{\alpha})\; n'_{\lambda}\; d\Sigma \\
&= \int J'^{\mu\nu 0}(t',\, \mathbf{x}')\; d^3x' \;.
\end{aligned}
\tag{2.63}
$$

Inserting Eq. 2.45 for the angular momentum current $J^{\mu\nu\lambda}$, we arrive at:

$$
J'^{\mu\nu}_{\mathrm{R}}(t') = \int \left(x'^{\mu}\, T'^{\nu 0}(t',\, \mathbf{x}') - x'^{\nu}\, T'^{\mu 0}(t',\, \mathbf{x}') \right) d^3x' \;. \tag{2.64}
$$

Since we are assuming that the system is static, we have $\partial'_0 T'^{\mu\nu} = 0$ and $T'^{i0} = 0$ (see Eq. 2.8). Eq. 2.64 then reduces to

$$
J'^{\mu\nu}_{\mathrm{R}}(t') = \int \left(x'^{\mu}\, \delta^{\nu}_{0} - x'^{\nu}\, \delta^{\mu}_{0} \right) T'^{00}(\mathbf{x}') d^3x'. \tag{2.65}
$$

The only possible time dependence of this expression can come from the factor in parentheses for the index combinations ($\mu = 0, \nu = i$) and ($\mu = i, \nu = 0$). One readily convinces oneself that even for these index combinations, the factor in parentheses does not depend on t'. So, for static systems $J'^{\mu\nu}_{\mathrm{R}}(t')$ does not depend on t', which means that $J^{\mu\nu}_{\mathrm{R}}(t)$ does not depend on t. This, in turn, means that the electromagnetic and non-electromagnetic static open sub-systems of the complete static system in the Trouton-Noble experiment do not give any turning couples. The delicately balanced turning couples we found on the basis of the Laue definition for the angular momentum of spatially extended systems are artifacts of the convention for choosing spacelike hyperplanes in that definition.

To conclude this section, I will prove one more result that we will need in section 2.3. I will show that in the Rohrlich picture, as in the Laue picture, the turning couple on static systems can be written as minus the cross product of the system's velocity and its momentum (cf. Eq. 2.60). To this end, I will prove that

$$
\tau^{ij}_{\mathrm{R}} = -\,\varepsilon^{ijk} \left(\mathbf{v} \times c\mathbf{P}_{\mathrm{R}} \right)_k, \tag{2.66}
$$

where $c\mathbf{P}_{\mathrm{R}}$ is the spatial part of the four-momentum under the Rohrlich definition. Inserting Eq. 2.64 into Eq. 2.62, we find

$$J_{\mathrm{R}}^{\mu\nu}(t) = \Lambda^{\mu}{}_{\rho}\,\Lambda^{\nu}{}_{\sigma} \int \left(x'^{\rho}T'^{\sigma 0} - x'^{\sigma}T'^{\rho 0}\right) d^3x' \,. \tag{2.67}$$

Since, as we have just seen (in Eq. 2.65), the integrand in this expression is independent of t' for static systems, we can invoke the result in Eq. 2.31 (cf. Fig. 2.3) and substitute γd^3x for d^3x'. If we further substitute $x^{\mu} = \Lambda^{\mu}{}_{\rho}x'^{\mu}$ and $x^{\nu} = \Lambda^{\nu}{}_{\sigma}x'^{\sigma}$, Eq. 2.67 turns into

$$J_{\mathrm{R}}^{\mu\nu}(t) = \gamma \int \left(x^{\mu}\left(\Lambda^{\nu}{}_{\sigma}T'^{\sigma 0}\right) - x^{\nu}\left(\Lambda^{\mu}{}_{\rho}T'^{\rho 0}\right)\right) d^3x \,. \tag{2.68}$$

Consider the ij-components of this equation and introduce the vector $c\mathbf{p}$ with components

$$cp^{i} \equiv \gamma\Lambda^{i}{}_{\rho}T'^{\rho 0}. \tag{2.69}$$

Inserting Eq. 2.69 into Eq. 2.68, we find

$$J_{\mathrm{R}}^{ij}(t) = \int c\left(x^{i}p^{j} - x^{j}p^{i}\right) d^3x \,. \tag{2.70}$$

In analogy with Eqs. 2.55–2.57, this can be rewritten as

$$J_{\mathrm{R}}^{ij}(t) = \int \varepsilon^{ijk}\left(\mathbf{x} \times c\mathbf{p}\right)_k d^3x \,. \tag{2.71}$$

The ij-components of the turning couple $\tau_{\mathrm{R}}^{\mu\nu}$ (cf. Eq. 2.50) can be written as

$$\tau_{\mathrm{R}}^{ij} = -\frac{dJ_{\mathrm{R}}^{ij}}{dt} = -\frac{d}{dt}\int \varepsilon^{ijk}\left(\mathbf{x} \times c\mathbf{p}\right)_k d^3x \,. \tag{2.72}$$

In analogy with Eq. 2.60 and Eqs. 1.58–Eq. 1.62 in section 1.4, Eq. 2.72 can be rewritten as

$$\tau_{\mathrm{R}}^{ij} = -\,\varepsilon^{ijk}\left(\mathbf{v} \times c\mathbf{P}\right)_k , \tag{2.73}$$

where $\mathbf{P}$ is defined as the space integral over $\mathbf{p}$. As the notation I chose suggests, $c\mathbf{P}$ is equal to $c\mathbf{P}_{\mathrm{R}}$, the spatial part of the four-momentum of the system under the Rohrlich definition. The proof of this identity involves yet another application of the result stated in Eq. 2.31 and illustrated in Fig. 2.3:

$$cP^{i} = \int \gamma\Lambda^{i}{}_{\rho}T'^{\rho 0}\, d^3x = \gamma\,\Lambda^{i}{}_{\rho}\,\frac{1}{\gamma}\int T'^{\rho 0}\, d^3x' = \Lambda^{i}{}_{\rho}P'^{\rho}_{\mathrm{R}} = P^{i}_{\mathrm{R}} \tag{2.74}$$

This concludes the proof of the claim in Eq. 2.66. In analogy with Eq. 2.58, we can introduce a turning couple $\mathbf{T}^{\mathrm{R}}$ via the relation

$$\tau_{\mathrm{R}}^{ij} \equiv c\, \varepsilon^{ijk}\, T_k^{\mathrm{R}}. \qquad (2.75)$$

Combining Eqs. 2.73–2.75, we can write

$$\mathbf{T}_{\mathrm{R}} = -\,\mathbf{v} \times \mathbf{P}_{\mathrm{R}}. \qquad (2.76)$$

So, both in the Laue picture (cf. Eq. 2.60) and in the Rohrlich picture, we can compute the turning couple on a static system simply by taking minus the cross product of its velocity and its momentum. We saw earlier that, in the Rohrlich picture, angular momentum is conserved for static systems, so that one never gets a turning couple on static systems. One can also infer this directly from Eq. 2.76. The velocity $\mathbf{v}$ and (c times) the momentum $\mathbf{P}_{\mathrm{R}}$ are the spatial parts of the system's four-velocity u^{μ} and its four-momentum P_{R}^{μ}, respectively. Under the Rohrlich definition, four-momentum transforms as a four-vector. For a static system in its rest frame, only the 0-component of the four-momentum is non-zero. So, in the system's rest frame, and therefore in any other frame, the four-momentum has the same direction as the four-velocity. Hence, the cross product of $\mathbf{v}$ and $\mathbf{P}_{\mathrm{R}}$ automatically vanishes for static systems.

2.3 Two 'four-momentum'-accounts of the Trouton-Noble experiment based on the Laue and Rohrlich definitions of four-momentum

2.3.1 Laue on the Trouton-Noble experiment as an example of a complete static system (1911). The discussion in sections 2.1 and 2.2 provides all the background necessary to read the dense final paragraph of Laue's 1911 paper "On the dynamics of relativity theory," in which he presented his account of the Trouton-Noble experiment for the first time.

> Another example of a complete static system [the first and only other example Laue gives is that of the Lorentz-Poincaré electron] is the condenser of the Trouton-Noble experiment with its field. The system as a whole does not need a turning couple when it has a uniform velocity, just as a mass point does not need one. The turning couple which the electromagnetic forces exert on the condenser is precisely the turning couple that the condenser, being an elastically stressed body, needs according to section 4 [i.e., the section where Laue introduces this peculiar effect in relativistic mechanics]. Neither the electromagnetic momentum, nor the mechanical momentum of the body have the direction of the velocity in this case, but the total momentum, the sum of the two of them, does, as follows from Eq. (22) [i.e., the equation expressing that the energy and momentum of a complete static system transform as a four-vector]. (Laue 1911a, p. 153)

So, the Laue picture of what happens in the moving condenser in the Trouton-Noble experiment is as follows. Neither the momentum $\mathbf{P}^{\mathrm{L}}_{\mathrm{EM}}$ of the electromagnetic part of the system, nor the momentum $\mathbf{P}^{\mathrm{L}}_{\text{non-EM}}$ of the remainder of the system[48] have the direction of the system's velocity $\mathbf{v}$. From the equation $\mathbf{T} = -\mathbf{v} \times \mathbf{P}$, it thus follows that there will be turning couples $\mathbf{T}^{\mathrm{L}}_{\mathrm{EM}}$ and $\mathbf{T}^{\mathrm{L}}_{\text{non-EM}}$. These turning couples will have to cancel one another, since the total momentum $\mathbf{P}^{\mathrm{L}}_{\mathrm{tot}} = \mathbf{P}^{\mathrm{L}}_{\mathrm{EM}} + \mathbf{P}^{\mathrm{L}}_{\text{non-EM}}$ of the complete static system is in the direction of $\mathbf{v}$, which means that $\mathbf{T}^{\mathrm{L}}_{\mathrm{tot}} = -\mathbf{v} \times \mathbf{P}^{\mathrm{L}}_{\mathrm{tot}}$ vanishes.

The Rohrlich picture of what happens in the moving condenser is not nearly as eventful. All momentum—$\mathbf{P}^{\mathrm{R}}_{\mathrm{EM}}, \mathbf{P}^{\mathrm{R}}_{\text{non-EM}}$, and $\mathbf{P}^{\mathrm{R}}_{\mathrm{tot}}$—is in the direction of the condenser's velocity $\mathbf{v}$, and there are no turning couples whatsoever.

2.3.2 Comparing the Laue picture of what happens in a moving condenser to the Rohrlich picture. Laue did not bother, neither in this paper nor in any of his later discussions of the Trouton-Noble experiment, to explicitly calculate the two separate contributions to the total momentum of the system that he distinguishes in the passage I quoted above. In order to bring out the strong contrast between the Laue picture and the Rohrlich picture of what happens

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[48] It is only as a matter of convenience that I refer to the remainder of the system as the 'non-electromagnetic part' of the system. What I call the 'electromagnetic part' of the system is actually just the *purely* electromagnetic part that depends only on the electromagnetic field; the 'non-electromagnetic part' includes the electric charge and current density, which (from a modern perspective) will be coupled to the electromagnetic field and to other matter fields. I am grateful to Tony Duncan for clarifying this point.

in the moving condenser in the Trouton-Noble experiment, I will derive expressions for $\mathbf{P}_{EM}$ and $\mathbf{P}_{non\text{-}EM}$ for this special case, both under the Laue and under the Rohrlich definition.

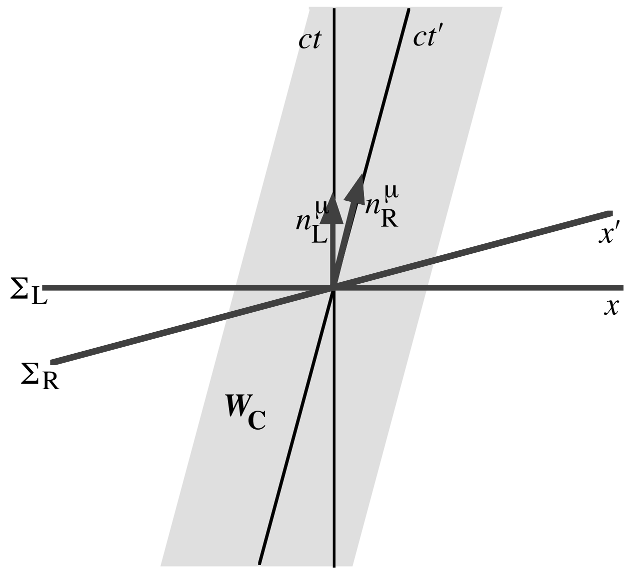

Figure 2.5 The hyperplanes Σ_L (Laue) and Σ_R (Rohrlich).

Fig. 2.5 shows a Minkowski diagram for the system I will consider. The shaded region W_C represents the bundle of worldlines (hence 'W') of all points inside a charged condenser (hence the subscript 'C') moving at a constant velocity $\mathbf{v}$ in the x-direction of a Lorentz frame with coordinates x^μ. The figure also shows a rest frame of the condenser with coordinates x'^μ. For the time being, we need not concern ourselves with the spatial geometry of the situation. Suffice it to say that I will consider the exact same situation I used for the calculations in chapter one (see, e.g., Fig. 1.5, Fig. 1.7, Fig. 1.9, and Fig 2.6 below). The task before us is to evaluate the various contributions to the four-momentum of this system according to both the Laue and the Rohrlich definition.

In this particular case, the Laue definition of the four-momentum P_L^μ in the x^μ-frame gives (cf. Eq. 2.42):

$$P_L^\mu = \int T^{\mu 0}\, d^3x = \int_{\Sigma_L} T^{\mu\nu} n_\nu^L \, d\Sigma, \qquad \text{[Laue]} \qquad (2.77)$$

with $n_L^\mu \equiv (1,\ 0,\ 0,\ 0)$, so that Σ_L is a hyperplane of simultaneity in the x^μ-frame (cf. Fig. 2.5).

For this same situation, the Rohrlich definition of the four-momentum P_R^μ in the x^μ-frame gives (cf. Eq. 2.43):

$$P^{\mu}_{\mathrm{R}} = \int_{\Sigma_{\mathrm{R}}} T^{\mu\nu} n^{\mathrm{R}}_{\nu}\, d\Sigma \qquad \text{[Rohrlich]} \qquad (2.78)$$

with $n^{\mu}_{\mathrm{R}} \equiv u^{\mu}/c$, where u^{μ} is the condenser's four-velocity, i.e., $u^{\mu} = \gamma\,(c,\ v, 0, 0)$. So, Σ_{R} is a hyperplane of simultaneity in the x'^{μ}-frame (cf. Fig. 2.5).

As will become clear below, the total energy-momentum tensor of the system as well as the two terms comprising this tensor have a very simple form. The components that are not zero everywhere, will at least always be constant inside the condenser, and zero outside. In other words, the integrands in Eq. 2.77 and Eq. 2.78 will be constant on the shaded region W_{C} in Fig. 2.5, and zero everywhere else. This means that the integrals can simply be replaced by the product of the integrands and the areas of the intersections of the respective hyperplanes Σ_{L} and Σ_{R} with the bundle of worldlines W_{C}. The area of a spacelike hyperplane can be interpreted as a spatial volume in one frame or another. Looking at Fig. 2.5 and recalling some basic facts about Lorentz transformations and Minkowski diagrams, one sees that the area of the intersection $\Sigma_{\mathrm{R}} \cap W_{\mathrm{C}}$ is just the volume V' of the condenser in the x'^{μ}-frame in which it is at rest; and that the area of the intersection $\Sigma_{\mathrm{L}} \cap W_{\mathrm{C}}$ is just the Lorentz contracted volume V'/γ of the condenser in the x^{μ}-frame in which it is moving. So, using the information that will be derived below about the form of the various terms in the system's energy-momentum tensor, we can replace the integral in Eq. 2.77, giving the four-momentum according to the Laue definition, by

$$P^{\mu}_{\mathrm{L}} = \left(T^{\mu\nu}_{\mathrm{inside}} n^{\mathrm{L}}_{\nu}\right) \mathsf{V}'/\gamma\ , \qquad \text{[Laue]} \qquad (2.79)$$

and the integral in Eq. 2.78, giving the four-momentum according to the Rohrlich definition, by

$$P^{\mu}_{\mathrm{R}} = \left(T^{\mu\nu}_{\mathrm{inside}} n^{\mathrm{R}}_{\nu}\right) \mathsf{V}'\ . \qquad \text{[Rohrlich]} \qquad (2.80)$$

For the factor in parentheses in Eq. 2.79 we can simply write

$$T^{\mu\nu}_{\mathrm{inside}} n^{\mathrm{L}}_{\nu} = T^{\mu 0}_{\mathrm{inside}} = \Lambda^{\mu}{}_{\rho}\, \Lambda^{0}{}_{\sigma}\, T'^{\rho\sigma}_{\mathrm{inside}}. \qquad \text{[Laue]} \qquad (2.81)$$

The factor in parentheses in Eq. 2.80 is slightly more complicated.[49] Using that $n^{\mathrm{R}}_{\mu} = \gamma(1, -\beta, 0, 0)$, we arrive at

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[49] It is only this particular way of calculating P^{μ}_{R} that appears to be more complicated than the corresponding calculation of P^{μ}_{L}. There obviously is a much simpler strategy for calculating the four-momentum in the x^{μ}-frame in the Rohrlich picture, a strategy that is not available in the Laue picture. Unlike P^{μ}_{L}, P^{μ}_{R} transforms as a four-vector, so we can simply compute P'^{μ} in the x'^{μ}-frame via $P'^{\mu}_{\mathrm{R}} = T^{\mu 0}_{\mathrm{inside}} \mathsf{V}'$ and transform to the x^{μ}-

$$T^{\mu\nu}_{\text{inside}} n^{\text{R}}_{\nu} = \gamma\, T^{\mu 0}_{\text{inside}} - \gamma\beta\, T^{\mu 1}_{\text{inside}}$$
[Rohrlich] (2.82)
$$= \gamma \Lambda^{\mu}{}_{\rho} \left(\Lambda^{0}{}_{\sigma} - \beta \Lambda^{1}{}_{\sigma} \right) T'^{\rho\sigma}_{\text{inside}}.$$

So, in order to evaluate the various contributions to P^{μ}_{L} or P^{μ}_{R}, we need expressions for the Lorentz transformation matrix $\Lambda^{\mu}{}_{\nu}$ and for the components $T'^{\mu\nu}$ of the various contributions to the system's energy-momentum in its rest frame.[50]

2.3.3 Derivation of expressions for the matrix of the Lorentz transformation to a conveniently chosen rest frame and for the electromagnetic and non-electromagnetic energy-momentum tensors in that frame.

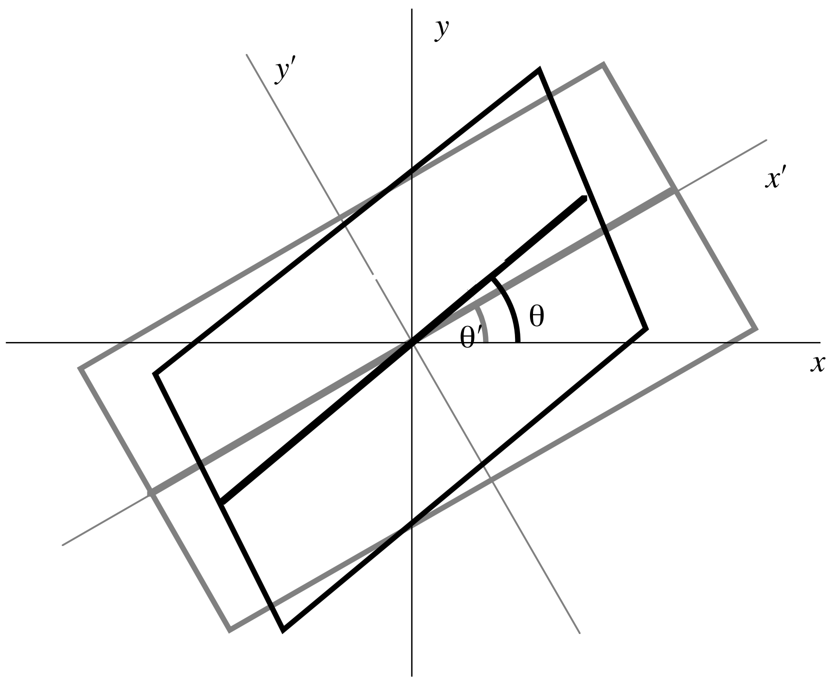

Figure 2.6 Condenser in the x^{μ}-frame in which it is moving (solid lines) and in the x'^{μ}-frame in which it is at rest (shaded lines).

The various terms in $T'^{\mu\nu}$ are most conveniently evaluated in an x'^{μ}-frame in which the plates of the condenser are perpendicular to one of the coordinate axes. In the situation I am interested in, however, the plates move at an arbitrary angle with respect to their velocity. Of course, we can

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frame using $P^{\mu}_{\text{R}} = \Lambda^{\mu}{}_{\nu} P'^{\nu}_{\text{R}}$. Rather than availing myself of this shortcut, I will go through the counterpart in the Rohrlich picture of the simplest derivation in the Laue picture. This will bring out the difference between the two accounts of the Trouton-Noble experiment in a more striking manner.

[50] Without loss of continuity, the reader can skip the next subsection in which these expressions are derived. The results can be found in Eq. 2.84 ($\Lambda^{\mu}{}_{\nu}$), Eqs. 2.88–2.89 ($T'^{\mu\nu}_{\text{EM}}$), and Eq. 2.96 ($T'^{\mu\nu}_{\text{non-EM}}$).

still evaluate $T'^{\mu\nu}$ in a convenient x'^{μ}-frame, as long as we take into account that, if we do so, the matrix $\Lambda^{\mu}{}_{\nu}$ in Eqs. 2.81–2.82 will represent a Lorentz transformation which is a combination of a spatial rotation and a boost. I will proceed in just this manner. The situation is illustrated in Fig. 2.6.

In the x'^{μ}-frame, the condenser has the familiar rectangular shape with plates of area $A = ab$, a distance d apart, and perpendicular to the y'-axis. In the x^{μ}-frame it has undergone a contraction by a factor of γ in the x-direction, the direction of motion. As in chapter one, I will have the top plate carry a positive charge Q and the bottom plate an equal but opposite charge $-Q$.

To get from the primed to the unprimed frame, we have to rotate the primed frame *clockwise* around the z′-axis over an angle θ', and then set it in motion in the (new) direction of the *negative* x'-axis at a velocity v. This passive transformation is equivalent to the following active transformation. Instead of looking upon the primed and unprimed quantities as describing the same situation in terms of different coordinates, look upon them as describing different situations in terms of the same coordinates. In particular, look upon the primed quantities as describing a new situation in terms of the unprimed coordinates This new situation is one in which the condenser is at rest with respect to the unprimed frame, its plates perpendicular to the y-axis. To get from this situation to the situation we have been looking at so far (with the condenser moving with respect to the unprimed frame) we have to rotate the condenser (rather than the frame) *counterclockwise* around the z-axis over an angle θ' and then set it in motion in the *positive* x-direction at a velocity v. The matrix $\Lambda^{\mu}{}_{\nu}$ for this transformation, interpreted actively or passively, is given by the matrix product

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta' & -\sin\theta' & 0 \\ 0 & \sin\theta' & \cos\theta' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{2.83}$$

which gives

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & \gamma\beta\cos\theta' & -\gamma\beta\sin\theta' & 0 \\ \gamma\beta & \gamma\cos\theta' & -\gamma\sin\theta' & 0 \\ 0 & \sin\theta' & \cos\theta' & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.84}$$

All that is left to do, now that we know what to insert for $\Lambda^{\mu}{}_{\nu}$ in Eqs. 2.81 and 2.82, is to compute the components of the various terms in the energy-momentum tensor for the system in the x'^{μ}-frame.[51] To begin with, I will divide the total energy-momentum tensor into two parts, a purely electromagnetic part and the rest. For convenience, I will refer to these parts as the electromagnetic and the non-electromagnetic part of the system, respectively:[52]

$$T'^{\mu\nu}_{\text{total}} = T'^{\mu\nu}_{\text{EM}} + T'^{\mu\nu}_{\text{non-EM}} \tag{2.85}$$

Since the system is complete and static, we have (cf. Eq. 2.6 and Eq. 2.8):

$$\begin{aligned} &\partial'_{\mu} T'^{\mu\nu}_{\text{tot}} = 0, && \text{[complete]} \\ \partial'_0 T'^{\mu\nu} = 0, \qquad &T'^{i0} = T'^{0i} = 0, && \text{[static]} \end{aligned} \tag{2.86}$$

where $T'^{\mu\nu}$ on the last line can be any of the three energy-momentum tensors in Eq. 2.85. These conditions allow us to determine $T'^{\mu\nu}_{\text{non-EM}}$ (with the exception of its 00-component) once we know $T'^{\mu\nu}_{\text{EM}}$, without having to know anything about the details of the system for which $T'^{\mu\nu}_{\text{non-EM}}$ is the energy-momentum tensor. This system can be any physical structure that prevents the charges and the plates carrying these charges from moving under the influence of the Coulomb interaction.

In order for the integrals in Eqs. 2.77–2.78 to reduce to the simple multiplication in Eqs. 2.79–2.80, I need two extra assumptions. First, I will assume that edge effects can be ignored, both for the electromagnetic and for the non-electromagnetic part of the system. Secondly, I will assume a simple form for $T'^{00}_{\text{non-EM}}$. Both assumptions, I claim, are totally innocuous.

To ensure that electromagnetic edge effects can safely be ignored, I assume that both a and b, determining the area $A = ab$ of the plates, are much larger than the distance d between the plates. This guarantees that I do not have to worry about inhomogeneities of the electromagnetic field at the edges of the plates. With this proviso, the electromagnetic field can simply be taken to vanish outside the condenser and to be homogeneous inside. In the x'^{μ}-frame, it is given by

$$\mathbf{E}' = (0, -E', 0), \qquad \mathbf{B}' = 0. \tag{2.87}$$

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[51] I am grateful to Tony Duncan for suggesting a particularly simple way of doing these calculations.

[52] This terminology is less arbitrary than it may seem. We will see in section 2.4 that the energy-momentum tensor for the electromagnetic part of the system gives the Coulomb forces on the condenser, whereas the energy-momentum tensor of the non-electromagnetic part gives the forces balancing these Coulomb forces.

Inserting Eq. 2.87 into Eqs. 2.3–2.5 for the components of the electromagnetic energy-momentum tensor $T'^{\mu\nu}_{\text{EM}}$, and setting the combination of constants $\frac{1}{2}\varepsilon_0 E'^2$ equal to u', we find:

$$T'^{\mu\nu}_{\text{EM}} = f(x',y',z')\ \text{diag}(u',\ u',-u',\ u'), \tag{2.88}$$

where the function $f(x',y',z')$ is equal to 1 inside the condenser and 0 outside. Using the step function $\theta(x)$,[53] we can write $f(x',y',z')$ as:

$$f(x',y',z') = \theta\left(\frac{a}{2} - |\,x'\,|\right)\theta\left(\frac{d}{2} - |\,y'\,|\right)\theta\left(\frac{b}{2} - |\,z'\,|\right). \tag{2.89}$$

One readily verifies that the electromagnetic part of the system, taken by itself, is not a complete system:

$$\partial'_\mu T'^{\mu\nu}_{\text{EM}} = \delta^{1\nu}\frac{\partial T'^{11}_{\text{EM}}}{\partial x'} + \delta^{2\nu}\frac{\partial T'^{22}_{\text{EM}}}{\partial y'} + \delta^{3\nu}\frac{\partial T'^{33}_{\text{EM}}}{\partial z'} \neq 0. \tag{2.90}$$

To be sure, $\partial'_\mu T'^{\mu\nu}_{\text{EM}}$ vanishes almost everywhere, but not on the surface forming the boundary between the inside and the outside of the condenser.

I will now determine the energy-momentum tensor $T'^{\mu\nu}_{\text{non-EM}}$ for the remainder of the system, i.e., the part that ensures that $\partial'_\mu T'^{\mu\nu}_{\text{tot}} = \partial'_\mu(T'^{\mu\nu}_{\text{EM}} + T'^{\mu\nu}_{\text{non-EM}}) = 0$. Since the system is static, $T'^{\mu\nu}_{\text{non-EM}}$ will have the form:

$$T'^{\mu\nu}_{\text{non-EM}} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & q & a & b \\ 0 & c & r & d \\ 0 & e & f & s \end{pmatrix}, \tag{2.91}$$

where all components can be functions of (x', y', z'). Since both $T'^{\mu\nu}_{\text{tot}} \to 0$ and $T'^{\mu\nu}_{\text{EM}} \to 0$ faster than $1/r'^3$ as $r' \to \infty$, it follows that these functions will have to go to zero faster than $1/r'^3$ as $r' \to \infty$, as well. From the symmetry of the situation, it follows that all off-diagonal components will be zero.[54] So, Eq. 2.91 reduces to:

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[53] The step function $\theta(x)$ is defined as follows: for $x<0$, $\theta(x)=0$; for $x=0$, $\theta(x)=1/2$; for $x>0$, $\theta(x)=1$.

[54] Looking back at Fig. 2.6, one sees that the experimental setup is symmetric under reflection in the $y'z'$-plane (i.e., changing x' to $-x'$) and under reflection in the $x'y'$-plane (i.e., changing z' to $-z'$). The matrices for the corresponding improper Lorentz transformations are:

$$\tilde{\Lambda}^\mu{}_\nu = \text{diag}(1, -1, 1, 1), \qquad \hat{\Lambda}^\mu{}_\nu = \text{diag}(1, 1, 1, -1).$$

Under these reflections, $T'^{\mu\nu}_{\text{non-EM}}$ transforms as follows:

$$T'^{\mu\nu}_{\text{non-EM}} = \text{diag}(p,\ q,\ r,\ s). \tag{2.92}$$

The functions q, r, and s can be determined from the conditions in Eq. 2.86. Consider the $\nu = 1$ component of $\partial'_\mu T'^{\mu\nu}_{\text{tot}} = 0$ (cf. Eq. 2.90):

$$\partial'_1\left(T'^{11}_{\text{EM}} + T'^{11}_{\text{non-EM}}\right) = 0. \tag{2.93}$$

It follows that

$$q = T'^{11}_{\text{non-EM}} = -T'^{11}_{\text{EM}} + C(y',z') = -u' f(x',y',z') + C(y',z'). \tag{2.94}$$

A simple argument shows that $C(y',z')$ has to be zero.[55] The 22- and 33-components of $T'^{\mu\nu}_{\text{non-EM}}$ can be found in exactly the same way. The 00-component can be any function of (x', y', z') that drops off faster than $1/r'^3$ as $r' \to \infty$. I will assume, for purposes of convenience, that

$$T'^{00}_{\text{non-EM}} = f(x',y',z')\, w', \tag{2.95}$$

where w' is an arbitrary constant. So, the final result is:[56]

$$\tilde{T}^{\mu\nu}_{\text{non-EM}} = \tilde{\Lambda}^{\mu}{}_{\rho}\, \tilde{\Lambda}^{\nu}{}_{\sigma}\, T'^{\rho\sigma}_{\text{non-EM}} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & q & -a & -b \\ 0 & -c & r & d \\ 0 & -e & f & s \end{pmatrix},\quad \hat{T}^{\mu\nu}_{\text{non-EM}} = \hat{\Lambda}^{\mu}{}_{\rho}\, \hat{\Lambda}^{\nu}{}_{\sigma}\, T'^{\rho\sigma}_{\text{non-EM}} = \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & q & a & -b \\ 0 & c & r & -d \\ 0 & -e & -f & s \end{pmatrix}$$

Since the symmetry of the experimental setup requires that

$$\tilde{T}^{\mu\nu}_{\text{non-EM}} = \hat{T}^{\mu\nu}_{\text{non-EM}} = T'^{\mu\nu}_{\text{non-EM}},$$

it follows that the off-diagonal ij-components of $T'^{\mu\nu}_{\text{non-EM}}$ are all zero: $a = b = c = d = e = f = 0$.

[55] Suppose $C(y',z') \neq 0$ for some values for y' and z'. Keep y' and z' fixed at those values and let x' go to infinity. Since $C(y',z')$ does not depend on x', it will remain non-zero. But this contradicts the assumption that $T'^{\mu\nu}_{\text{non-EM}} \to 0$ for $r' \to \infty$. Hence, $C(y',z') = 0$.

[56] Adding Eq. 2.96 for $T'^{\mu\nu}_{\text{non-EM}}$ to Eq. 2.88 for $T'^{\mu\nu}_{\text{EM}}$, we find that the total energy-momentum tensor in the system's rest frame is given by:

$$T'^{\mu\nu}_{\text{tot}} = T'^{\mu\nu}_{\text{EM}} + T'^{\mu\nu}_{\text{non-EM}} = f(x',y',z')\ \text{diag}(u'+w',\ 0,\ 0,\ 0).$$

Notice that the ij-components of $T'^{\mu\nu}_{\text{tot}}$, representing the stresses in the rest frame, all vanish. This is in accordance with "Laue's theorem" (see Eqs. 2.35–2.38), which says that the integrals over the stresses in the rest frame all ought to vanish for a complete static system.

$$T'^{\mu\nu}_{\text{non-EM}} = f(x',y',z')\ \text{diag}(w',-u',\ u',-u'). \tag{2.96}$$

2.3.4 The division of the total momentum into electromagnetic and non-electromagnetic momentum in the Laue and in the Rohrlich picture. We now have all ingredients to compute the various contributions to the four-momentum of the condenser plus its field in the Trouton-Noble experiment, according to the definitions of Laue and Rohrlich, respectively.

Like the total energy-momentum tensor, I divide the total four-momentum in the system into an electromagnetic and a non-electromagnetic part

$$P^{\mu}_{\text{tot}} = P^{\mu}_{\text{EM}} + P^{\mu}_{\text{non-EM}}. \tag{2.97}$$

The total four-momentum is the same no matter whether we use Laue's or Rohrlich's definition. In the x'^{μ}-frame, it is given by (cf. Eq. 2.88 and Eq. 2.96)

$$P'^{\mu}_{\text{tot}} = T'^{\mu 0}_{\text{tot}_{\text{inside}}}\, V' = (U'_{\text{tot}},\ 0,\ 0,\ 0), \tag{2.98}$$

where $U'_{\text{tot}} \equiv U'_{\text{EM}} + U'_{\text{non-EM}}$, with $U'_{\text{EM}} = u'V'$ and $U'_{\text{non-EM}} = w'V'$.[57] This is a four-vector, so in the x^{μ}-frame, it is given by

$$P^{\mu}_{\text{tot}} = \Lambda^{\mu}{}_{\nu}\, P'^{\nu}_{\text{tot}} = (\gamma U'_{\text{tot}},\ \gamma\beta U'_{\text{tot}}, 0, 0)\ . \tag{2.99}$$

How the total four-momentum breaks up into an electromagnetic and a non-electromagnetic part depends on whether we use the Laue or the Rohrlich definition of four-momentum, *unless* we are in the system's rest frame, where the hyperplanes Σ_L and Σ_R that distinguish these two definitions coincide (cf. Fig. 2.5). Hence, in the x'^{μ}-frame, we have (see Eqs. 2.79–2.80 for P^{μ}_{L} and P^{μ}_{R}):

$$\begin{aligned} P_L{}'^{\mu}_{\text{EM}} &= P_R{}'^{\mu}_{\text{EM}} = T'^{\mu 0}_{\text{EM}}\, V' = (U'_{\text{EM}},\ 0,\ 0,\ 0),\\ P_L{}'^{\mu}_{\text{non-EM}} &= P_R{}'^{\mu}_{\text{non-EM}} = T'^{\mu 0}_{\text{non-EM}}\, V' = (U'_{\text{non-EM}},\ 0,\ 0,\ 0). \end{aligned} \tag{2.100}$$

I will evaluate the electromagnetic and the non-electromagnetic four-momentum in the x^{μ}-frame, first using the Laue then the Rohrlich definition.

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[57] In the remainder of this section, I will suppress the subscript 'inside.' Whenever an energy-momentum tensor occurs in the equations below (Eqs. 2.99–2.115), I actually mean the constant values of that tensor in the shaded region W_C in Fig. 2.5.

Under the Laue definition, we can write the total four-momentum in the x^{μ}-frame as (see Eq. 2.79 and Eq. 2.81)

$$\begin{aligned} P^{\mu}_{\text{tot}} &= P\text{L}^{\mu}_{\text{EM}} + P\text{L}^{\mu}_{\text{non-EM}} \\ &= \Lambda^{\mu}{}_{\rho}\, \Lambda^{0}{}_{\sigma} \left(T'^{\rho\sigma}_{\text{EM}} + T'^{\rho\sigma}_{\text{non-EM}}\right) \mathsf{V}'/\gamma. \end{aligned} \tag{2.101}$$

The fact that both $T'^{\mu\nu}_{\text{EM}}$ and $T'^{\mu\nu}_{\text{non-EM}}$ are diagonal considerably simplifies Eq. 2.101. Inserting Eq. 2.88 for $T'^{\mu\nu}_{\text{EM}}$ and Eq. 2.96 for $T'^{\mu\nu}_{\text{non-EM}}$, we find the following expressions for $P\text{L}^{\mu}_{\text{EM}}$ and $P\text{L}^{\mu}_{\text{non-EM}}$:

$$P\text{L}^{\mu}_{\text{EM}} = \frac{1}{\gamma} \left(\Lambda^{\mu}{}_{0}\, \Lambda^{0}{}_{0} + \Lambda^{\mu}{}_{1}\, \Lambda^{0}{}_{1} - \Lambda^{\mu}{}_{2}\, \Lambda^{0}{}_{2}\right) U'_{\text{EM}}\,, \tag{2.102}$$

$$P\text{L}^{\mu}_{\text{non-EM}} = \frac{1}{\gamma}\, \Lambda^{\mu}{}_{0}\, \Lambda^{0}{}_{0}\, U'_{\text{non-EM}} + \frac{1}{\gamma} \left(- \Lambda^{\mu}{}_{1}\, \Lambda^{0}{}_{1} - \Lambda^{\mu}{}_{2}\, \Lambda^{0}{}_{2}\right) U'_{\text{EM}}\,, \tag{2.103}$$

where I used that $U'_{\text{EM}} = u'\mathsf{V}'$ and $U'_{\text{non-EM}} = w'\mathsf{V}'$, plus the fact that $\Lambda^{0}{}_{3} = 0$ (see Eq. 2.84). Using Eq. 2.84 for the remaining components of $\Lambda^{\mu}{}_{\nu}$, we find

$$P\text{L}^{\mu}_{\text{EM}} = U'_{\text{EM}} \begin{pmatrix} \gamma\,(1 + \beta^2 \cos^2\theta' - \beta^2 \sin^2\theta') \\ 2\gamma\beta\cos^2\theta' \\ 2\beta\sin\theta'\cos\theta' \\ 0 \end{pmatrix}, \tag{2.104}$$

and

$$P\text{L}^{\mu}_{\text{non-EM}} = U'_{\text{non-EM}} \begin{pmatrix} \gamma \\ \gamma\beta \\ 0 \\ 0 \end{pmatrix} + U'_{\text{EM}} \begin{pmatrix} \gamma\,(\beta^2 \sin^2\theta' - \beta^2 \cos^2\theta') \\ -\gamma\beta\cos^2\theta' + \gamma\beta\sin^2\theta' \\ -2\beta\sin\theta'\cos\theta' \\ 0 \end{pmatrix}. \tag{2.105}$$

Using that $U'_{\text{EM}} + U'_{\text{EM}} = U'_{\text{tot}}$, one easily verifies that adding Eq. 2.104 and Eq. 2.105 gives Eq. 2.99 above for P^{μ}_{tot} .

Consider the spatial parts of Eqs. 2.104–2.105. Recall that these components are equal to c times the ordinary momentum $\mathbf{P}$. The total (ordinary) momentum, both under the Laue and under the Rohrlich definition, is given by (see Eq. 2.99):

$$\mathbf{P}_{\text{tot}} = \gamma \left(U'_{\text{tot}}/c^2\right) \mathbf{v}\,, \tag{2.106}$$

where I wrote v/c for β and $\mathbf{v}$ for $(v, 0, 0)$. The quantity in parentheses can be identified with the total rest mass of the system.

From Eqs. 2.104–2.105, we can read off the electromagnetic and non-electromagnetic contributions to the total momentum, under the Laue definition:

$$\mathbf{P}^{\mathrm{L}}_{\mathrm{EM}} = 2\left(U'_{\mathrm{EM}}/c^2\right) v \cos\theta' \begin{pmatrix} \gamma\cos\theta' \\ \sin\theta' \\ 0 \end{pmatrix}, \tag{2.107}$$

and

$$\begin{aligned}\mathbf{P}^{\mathrm{L}}_{\text{non-EM}} = {} & \gamma\left(U'_{\text{non-EM}}/c^2\right) v \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \\ & + \left(U'_{\mathrm{EM}}/c^2\right) v \left[-\cos\theta' \begin{pmatrix} \gamma\cos\theta' \\ \sin\theta' \\ 0 \end{pmatrix} + \sin\theta' \begin{pmatrix} \gamma\sin\theta' \\ -\cos\theta' \\ 0 \end{pmatrix} \right].\end{aligned} \tag{2.108}$$

One easily verifies that adding Eq. 2.107 and Eq. 2.108 gives $\mathbf{P}_{\mathrm{tot}}$ in Eq. 2.106. Notice that Eq. 2.107 agrees with the result found by Lorentz (see section 1.4, Eq. 1.75) as was to be expected.[58]

To first order in β (which means $\gamma = 1$ and $\theta' = \theta$), the interpretation of the four terms in Eqs. 2.107–2.108 is very simple. As was already noticed by Lorentz (see section 1.4, Fig. 1.11, Eq. 1.36 and Eq. 1.67), $\mathbf{P}^{\mathrm{L}}_{\mathrm{EM}}$, proportional to $\cos\theta$, is parallel to the plates in this approximation. What about the three terms in $\mathbf{P}^{\mathrm{L}}_{\text{non-EM}}$? The first θ-independent term has the direction of the velocity $\mathbf{v}$; the second, proportional to $\cos\theta$ and parallel to the plates, cancels half of $\mathbf{P}^{\mathrm{L}}_{\mathrm{EM}}$; and the third, proportional to $\sin\theta$, is perpendicular to the plates. Fig. 2.7 shows the various terms in Eqs. 2.107–2.108 for the situation with $\beta \approx .75$ used in all figures for the moving condenser.

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[58] In relativistic terms, the derivation of Eq. 1.75 runs as follows: (a) transform the electromagnetic field from the x'^{μ}-frame to the x^{μ}-frame; (b) compute the energy-momentum tensor for this electromagnetic field in the x^{μ}-frame; (c) integrate the $\mu 0$-components of this energy-momentum tensor over a hyperplane of simultaneity in the x^{μ}-frame. The derivation of Eq. 2.107 runs as follows: (a′) compute the energy-momentum tensor for the electromagnetic field in the x'^{μ}-frame; (b′) transform this energy-momentum tensor from the x'^{μ}-frame to the x^{μ}-frame; (c′) integrate the $\mu 0$-components of this energy-momentum tensor over a hyperplane of simultaneity in the in the x^{μ}-frame. Obviously, (a)-(c) and (a′)-(c′) should give the same result. The 0-component of Eq. 2.104 for the electromagnetic four-momentum is likewise equal to Eq. 1.73 for the electromagnetic energy that we derived in the context of Lorentz's theory.

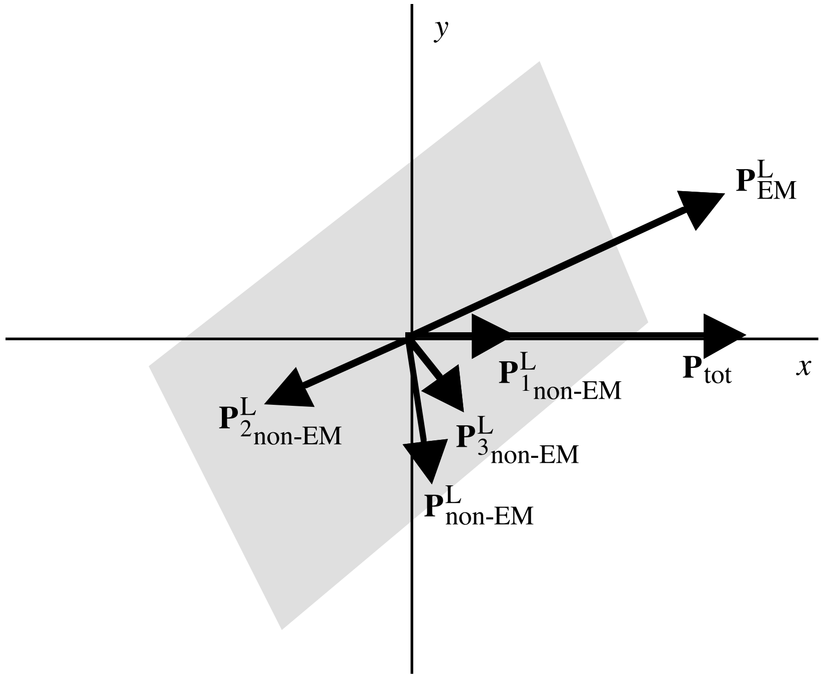

Figure 2.7 Electromagnetic and non-electromagnetic contributions to the total momentum in a charged moving condenser according to Laue.

Eqs. 2.107–2.108 for $\mathbf{P}^{\mathrm{L}}_{\mathrm{EM}}$ and $\mathbf{P}^{\mathrm{L}}_{\text{non-EM}}$ and Fig. 2.7 fill in the details of the picture Laue sketches in the passage I quoted at the beginning of this section: "Neither the electromagnetic momentum, nor the mechanical momentum of the body have the direction of the velocity in this case, but the total momentum, the sum of the two of them, does" (Laue 1911a, p. 153). Using the equation $\mathbf{T} = -\mathbf{v} \times \mathbf{P}$ (see section 1.4, Eq. 1.38 and Eq. 1.62), one easily calculates the turning couples coming from the various contributions to the momentum. Suffice it to say that the result for the electromagnetic turning couple coincides with the result found by Lorentz (see section 1.4, Eq. 1.39).

I want to draw attention to the fact that, under the Laue definition of four-momentum, the electromagnetic and non-electromagnetic parts of the condenser's momentum, like the electromagnetic part of the momentum of Lorentz-Poincaré electron, do not separately transform as the momentum of a relativistic point particle. In the case of the Trouton-Noble experiment, the deviations of $\mathbf{P}^{\mathrm{L}}_{\mathrm{EM}}$ and $\mathbf{P}^{\mathrm{L}}_{\text{non-EM}}$ in Eqs. 2.107–2.108 from the equations

$$\mathbf{P}_{\mathrm{EM}} = \gamma \left(U'_{\mathrm{EM}} / c^2 \right) \mathbf{v}, \qquad \mathbf{P}_{\text{non-EM}} = \gamma \left(U'_{\text{non-EM}} / c^2 \right) \mathbf{v}. \tag{2.109}$$

that one would have if the electromagnetic part of the condenser's momentum were to transform as the momentum of a relativistic point particle are far more severe than the puzzling factor $4/3$ in the case of the Lorentz-Poincaré electron (cf. Eqs. 2.39–2.40). Thanks to the spherical symmetry of the Lorentz-Poincaré electron in its rest frame, its electromagnetic momentum at least always has the direction of the electron's velocity. For the condenser in the Trouton-Noble experiment, as we have just seen, this is not the case.[59]

I will now go through the analogue in the Rohrlich picture of the derivation in Eqs. 2.101–2.108 in the Laue picture. Under the Rohrlich definition of four-momentum, we can write the total four-momentum in the x^μ-frame as (see Eq. 2.80 and Eq. 2.82)

$$\begin{aligned} P^\mu_{\text{tot}} &= P\text{R}^\mu_{\text{EM}} + P\text{R}^\mu_{\text{non-EM}} \\ &= \gamma\, \Lambda^\mu{}_\rho \left(\Lambda^0{}_\sigma - \beta\Lambda^1{}_\sigma\right)\left(T'^{\rho\sigma}_{\text{EM}} + T'^{\rho\sigma}_{\text{non-EM}}\right) V\,'. \end{aligned} \tag{2.110}$$

Consider the factor $\Lambda^0{}_\sigma - \beta\Lambda^1{}_\sigma$ in this equation. Inserting Eq. 2.84 for $\Lambda^\mu{}_\nu$, we find that

$$\begin{aligned} &\Lambda^0{}_0 - \beta\Lambda^1{}_0 = \gamma - \beta(\gamma\beta) = \frac{1}{\gamma}, \\ &\Lambda^0{}_1 - \beta\Lambda^1{}_1 = \gamma\beta\cos\theta' - \beta(\gamma\cos\theta') = 0, \\ &\Lambda^0{}_2 - \beta\Lambda^1{}_2 = -\gamma\beta\sin\theta' - \beta(-\gamma\sin\theta') = 0, \\ &\Lambda^0{}_3 - \beta\Lambda^1{}_3 = 0. \end{aligned} \tag{2.111}$$

As a consequence, the θ'-dependence responsible for the strange transformation behavior of the four-momentum of the condenser under the Laue definition completely disappears under the Rohrlich definition. Moreover, Eqs. 2.110–2.111 show that, under the Rohrlich definition of four-momentum, stresses in the x'^μ-frame do not contribute to the four-momentum in the x^μ-frame. Eq. 2.111 can be summarized as

$$\Lambda^0{}_\sigma - \beta\Lambda^1{}_\sigma = \frac{1}{\gamma}\,\delta^0_\sigma. \tag{2.112}$$

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[59] I would like to suggest that this may have contributed to the fact that the strange transformation behavior of the electromagnetic part of the four-momentum of the condenser in the Trouton-Noble experiment never gained the notoriety of the '$^4/_3$-puzzle' in the transformation behavior of the electromagnetic part of the four-momentum of the Lorentz-Poincaré electron. In the case of the electromagnetic part of the condenser in the Trouton-Noble experiment, the deviation from the behavior of a relativistic point mass is so strong that it never even becomes a puzzle!

Substituting this expression into Eq. 2.110, we find the following expressions for $P_{\mathrm{L}}{}^{\mu}_{\mathrm{EM}}$ and $P_{\mathrm{L}}{}^{\mu}_{\text{non-EM}}$:

$$P_{\mathrm{R}}{}^{\mu}_{\mathrm{EM}} = \Lambda^{\mu}{}_{\rho}\, T'^{\rho 0}_{\mathrm{EM}}\, V', \qquad P_{\mathrm{R}}{}^{\mu}_{\text{non-EM}} = \Lambda^{\mu}{}_{\rho}\, T'^{\rho 0}_{\text{non-EM}}\, V'. \tag{2.113}$$

The quantities $T'^{\rho 0}_{\mathrm{EM}}\, V'$ and $T'^{\rho 0}_{\text{non-EM}}\, V'$ in Eq. 2.113 are just $P'_{\mathrm{R}}{}^{\rho}_{\mathrm{EM}}$ and $P'_{\mathrm{R}}{}^{\rho}_{\text{non-EM}}$, the electromagnetic and non-electromagnetic parts of the system's four-momentum in the x'^{μ}-frame, respectively. Inserting Eq. 2.84 for $\Lambda^{\mu}{}_{\nu}$ and Eq. 2.100 for $P'_{\mathrm{R}}{}^{\rho}_{\mathrm{EM}}$ and $P'_{\mathrm{R}}{}^{\rho}_{\text{non-EM}}$ into Eq. 2.113, we arrive at:

$$P_{\mathrm{R}}{}^{\mu}_{\mathrm{EM}} = \Lambda^{\mu}{}_{\rho} P'_{\mathrm{R}}{}^{\mu}_{\mathrm{EM}} = \left(\gamma\, U'_{\mathrm{EM}}, \gamma\beta\, U'_{\mathrm{EM}}, 0, 0\right), \qquad P_{\mathrm{R}}{}^{\mu}_{\mathrm{nonEM}} = \Lambda^{\mu}{}_{\rho} P'_{\mathrm{R}}{}^{\mu}_{\mathrm{nonEM}} = \left(\gamma\, U'_{\text{non-EM}}, \gamma\beta\, U'_{\text{non-EM}}, 0, 0\right). \tag{2.114}$$

From Eq. 2.114 one can read off expressions for $\mathbf{P}^{\mathrm{R}}_{\mathrm{EM}}$ and $\mathbf{P}^{\mathrm{R}}_{\text{non-EM}}$, the electromagnetic and non-electromagnetic parts of the ordinary momentum of the system under the Rohrlich definition:

$$\mathbf{P}^{\mathrm{R}}_{\mathrm{EM}} = \gamma \left(U'_{\mathrm{EM}}/c^2\right) \mathbf{v}, \qquad \mathbf{P}^{\mathrm{R}}_{\text{non-EM}} = \gamma \left(U'_{\text{non-EM}}/c^2\right) \mathbf{v}. \tag{2.115}$$

Notice that $\mathbf{P}^{\mathrm{R}}_{\mathrm{EM}}$ and $\mathbf{P}^{\mathrm{R}}_{\text{non-EM}}$ are indeed equal to the expression for $\mathbf{P}_{\mathrm{EM}}$ and $\mathbf{P}_{\text{non-EM}}$ in Eq. 2.109, as was to be expected since under the Rohrlich definition, the four-momentum of any system is a four-vector. Since $\mathbf{P}^{\mathrm{R}}_{\mathrm{EM}}$ and $\mathbf{P}^{\mathrm{R}}_{\text{non-EM}}$ are in the direction of the velocity $\mathbf{v}$, the turning couples $\mathbf{T}^{\mathrm{R}}_{\mathrm{EM}} = -\,\mathbf{v} \times \mathbf{P}^{\mathrm{R}}_{\mathrm{EM}}$ and $\mathbf{T}^{\mathrm{R}}_{\text{non-EM}} = -\,\mathbf{v} \times \mathbf{P}^{\mathrm{R}}_{\text{non-EM}}$ will both be zero. In the Rohrlich picture, there are no turning couples in the Trouton-Noble experiment whatsoever.

2.3.5 A closer look at the difference between the Laue and Rohrlich picture of what happens in moving condensers: the argument for the kinematical nature of the Laue effect. Perhaps the best way to bring out the difference between the Laue picture and the Rohrlich picture of what happens in a charged moving condenser is to consider the following variation on the Trouton-Noble experiment.[60] Suppose we slowly change the angle θ', slowly so as to preserve the static character of the situation. Since the total energy is independent of

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[60] I will return to this variation on the Trouton-Noble experiment in section 4.2 (see Figs. 4.1–4.2).

θ' (see Eq. 2.99), this can be done with arbitrarily small expenditure of work on our part. In other words, we can slowly and adiabatically rotate the moving condenser around an axis perpendicular to its velocity and parallel to its plates. I want to compare the Laue picture of what happens in this experiment with the Rohrlich picture.

First of all, the shape of the condenser changes (cf. the discussion of Larmor's account of the Trouton-Noble experiment in section 1.3). This is understood to be a purely kinematical effect. In the Rohrlich picture, this is all that happens.[61] Not only will the total four-momentum and the total angular momentum be conserved, the electromagnetic and non-electromagnetic contributions to these quantities will be conserved separately as well. In the Laue picture, this is not the case. The total four-momentum and the total angular momentum are, of course, conserved, but the electromagnetic and non-electromagnetic contributions to these quantities considered separately are not. From Eqs. 2.104–2.105, one can read off that, as θ' goes from, say, 0 to $\pi/2$, energy and momentum flow from the electromagnetic to the non-electromagnetic part of the system. In terms of angular momentum, the situation is similar. Recall that the turning couple is equal to minus the rate of change of angular momentum (cf. Eq. 2.50). With $T^{\mathrm{L}}_{\mathrm{EM}_z} = -T^{\mathrm{L}}_{\text{non-EM}_z} \approx -U'_{\mathrm{EM}}\beta^2 \sin 2\theta$ (cf. section 1.4, Eq. 1.39), this means that, as θ' goes from, say, 0 to $\pi/4$, angular momentum flows from the non-electromagnetic to the electromagnetic part of the system.

Since we only find these manifestations of the Laue effect in the Laue picture and not in the Rohrlich picture,[62] it is clear that they are artifacts of the standard convention that Laue uses for choosing spacelike hyperplanes in the definition of the four-momentum and the angular momentum of spatially extended systems. This observation should not be construed as denying the reality of these manifestations of the Laue effect. I do not mean to imply that at all. The point is that the Laue effect is a kinematical rather than a dynamical effect. Its status is exactly the same as the status of the more familiar kinematical effects in special relativity, viz. length contraction and time dilation. Once we agree on the standard definition of 'the length of a moving rod' or 'the rate of a moving clock,' moving rods *are* shorter than rods at rest and clocks in motion *do* tick at a slower rate than clocks at rest. Still, we recognize that length contraction and time dilation do not tell us anything about the dynamics of the systems we use

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[61] As we will see shortly, even the length contraction effect can be defined away in the Rohrlich picture.

[62] The exchange of four-momentum and angular momentum in the x^{μ}-frame that we found under the Laue definition is directly related to the exchange of energy-momentum between the two constituents of the system in the x'^{μ}-frame, in which the system is at rest. This exchange of energy-momentum only involves the stress components of the energy-momentum tensors. Under the Rohrlich definition, these stress components do not give rise to momentum or angular momentum in the x^{μ}-frame. Hence, there will be no exchange of four-momentum and angular momentum in the x^{μ}-frame under the Rohrlich definition.

as clocks or rods, but only about their spatio-temporal behavior and about our conventions in calling one rather than another spacelike slice of the rod's bundle of worldlines 'the length of the moving rod,' and in calling one rather than another timelike slice of the clock's bundle of worldlines 'the rate of the moving clock.'

The situation with the Laue effect is fully analogous. Once we agree on the Laue definitions of four-momentum and angular momentum of spatially extended systems, the Laue effect and its manifestations in the experiment I just described are as tangible as any other effect. Still, we have to recognize that the effects only reflect the spatio-temporal behavior of the system under consideration and our convention of picking one rather than another spacelike slice of its bundle of worldlines in defining its four-momentum and angular momentum in a frame in which it is moving.

It can hardly be denied, I think, that both in the case of the actual Trouton-Noble experiment and in the case of the variation on it considered above, the Rohrlich picture of what happens fits much better with our physical intuitions than the Laue picture. After all, in a system that is moving uniformly, retaining static equilibrium while being slowly and adiabatically rotated in its rest frame, we would not expect its constituents to exchange momentum and angular momentum. The Rohrlich picture with no such exchanges is clearly more satisfactory. There is a price for this though. Recall that the physical interpretation of Rohrlich's definition of four-momentum is problematic (see the discussion following Eq. 2.43). What we mean by 'the four-momentum of a spatially extended system in some arbitrary x^μ-frame' would seem to be an ordinary space integral in the x^μ-frame over the system's four-momentum density and not, as Rohrlich wants to have it, some integral over a tilted hyperplane in the x^μ-frame.

It will be helpful to explore the analogy with length contraction in somewhat greater detail. Look back at Fig. 2.5 showing the bundle of worldlines W_C of a condenser at rest with respect to the x'^μ-frame and in uniform motion with respect to the x^μ-frame. How do we define 'the length of the condenser in the x^μ-frame?' Under the standard definition, 'the length of the condenser in the x^μ-frame' is the length of the intersection $\Sigma_L \cap W_C$, where Σ_L is a hyperplane of simultaneity in the x^μ-frame. Another option would be to define 'the length of the condenser in the x^μ-frame' as the length of the intersection $\Sigma_R \cap W_C$, where Σ_R is a hyperplane of simultaneity in the condenser's rest frame. One will recognize that the former definition is analogous to the Laue definitions of four-momentum and angular momentum in Eq. 2.42 and Eq. 2.48, while the latter definition is analogous to the Rohrlich definitions of those same quantities in Eq. 2.43 and Eq. 2.49. Under the Laue definition of 'the length of the condenser in the x^μ-frame,' the moving condenser will exhibit the length contraction effect; under the Rohrlich definition, it will not. Length contraction thus is an artifact of our convention of picking a spacelike hyperplane in defining the length of an object in some arbitrary x^μ-frame.

From this observation, I may add, we could infer, if we did not already realize this, that length contraction is a kinematical effect. The point I want to make here is that the standard frame-dependent Laue definition of length in some arbitrary x^μ-frame is perfectly reasonable. So, we have no trouble accepting the length contraction effect it entails, even though the effect is highly counter-intuitive. Likewise, it is perfectly reasonable to retain the standard frame-dependent Laue definitions of the four-momentum and angular momentum of spatially extended systems, and to accept the Laue effect these definitions entail, along with the counter-intuitive exchanges of four-momentum and angular momentum in the variant on the Trouton-Noble experiment I described.

Whether one adopts Laue definitions or Rohrlich definitions is, in the final analysis, a matter of convention. However, if we adopt the standard Laue definitions, we have to remember that effects such as length contraction, time dilation, and the Laue effect are all purely kinematical effects—they reflect the standard spatio-temporal behavior of systems in Minkowski space-time, our conventions about slicing space-time one way or another, and nothing else.

To conclude this section, I want to anticipate a possible objection to my analysis and respond to it. The most convincing way to argue for my claim that the Laue effect is purely kinematical is through the following modus ponens argument. If length contraction is purely kinematical, then the Laue effect is purely kinematical as well. Length contraction clearly is purely kinematical. Therefore, the Laue effect is also purely kinematical.

According to a well-known philosophical proverb, one man's modus ponens is another man's modus tollens. All the hard work that went into establishing the conditional premise of my argument would thus come to naught if this premise were to end up in the hands of those dissidents from relativistic orthodoxy who want to deny that length contraction is purely kinematical (see, e.g., Bell 1987,[63] Dieks 1984, Prokhovnik 1963). Since the Laue effect

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[63] In an interview published in 1986 (Davies and Brown 1986), Bell argues that the "cheapest solution" to the problems raised by the Einstein-Podolsky-Rosen argument, the Bell inequalities, and the Aspect experiments would be to go back to pre-relativistic ether theory (my favorite simple derivation of the quantum mechanical predictions for EPR-type experiments can be found in Albert 1992, Ch. 3, pp. 61–72; my favorite simple derivation of the Bell inequalities can be found in Penrose 1989, pp. 279–285). Bell went on record saying among other things: "I would say that the cheapest resolution is something like going back to relativity as it was before Einstein, when people like Lorentz and Poincaré thought that there was an aether—a preferred frame of reference—but that our instruments were distorted by motion in such a way that we could not detect motion through the aether [let me emphasize that Lorentz only started thinking along those lines, under Einstein's influence, *after 1905*]. Now, in that way you can imagine that there is a preferred frame of reference, and in this preferred frame of reference things do go faster than light. But then in other frames of reference when they seem to go not only faster than light but backwards in time, that is an optical illusion" (Davies and Brown 1986, p. 49). Bell adds: "what is not sufficiently emphasized in the textbooks, in my opinion, is that the pre-Einsteinian position of Lorentz and Poincaré, Larmor and FitzGerald was perfectly coherent, and is not inconsistent with relativity theory. The idea that there is an aether, and these FitzGerald contractions and Larmor dilations occur,

undoubtedly looks a lot less like a kinematical effect than length contraction, they might want to bolster their case with the following modus tollens argument. If length contraction is purely kinematical, then the Laue effect is purely kinematical as well. Clearly, the Laue effect is not purely kinematical. Therefore, as the dissident has been saying all along, length contraction is not purely kinematical either.

Why would anybody want to deny that length contraction is purely kinematical? I can think of several reasons, good and bad.[64] First, there is a sound pedagogical reason. By calling length contraction purely kinematical one runs the risk of perpetuating the persistent misconception, particularly prevalent among philosophically naive physics undergraduates, that length contraction is not a real effect. As I emphasized, kinematical effects are very real. A good way to drive home that point is to stress that it is a highly non-trivial empirical fact about our actual world that physical systems to a very good approximation exhibit the spatio-temporal behavior that results in us detecting the phenomenon of length contraction once we agree to use certain spacelike slices of the system to define its length in various Lorentz frames.

Does this last observation imply that length contraction is at least to some extent dynamical? That depends on one's definition of what counts as a purely kinematical effect. At first sight, it looks as if under the definition licensed by the dominant tradition in modern philosophy of space and time,[65] length contraction is at least partly dynamical. This first impression is misleading. In this modern tradition, all space-time theories are formulated as sets of models of the form $<M, O_i>$, where M is a differentiable manifold and where the O_i's are geometric object fields. Some of the O_i's encode the space-time structure, others describe the contents of space-time. The natural distinction between kinematics and dynamics in this approach seems to be the following. As long as we are talking only about those geometric object fields encoding the space-time structure, we are talking kinematics, the moment we start talking about geometric object fields describing the contents of space-time, we are talking dynamics. If we understand length contraction to be part of the spatio-temporal behavior of systems in a Minkowski space-time, as I want to do, we are definitely talking about more than just the geometric object fields encoding the space-time structure and we would therefore seem to be forced to accept that length contraction is at least partly dynamical. This argument is flawed. It conflates ontology with epistemology.[66] It is true that we need physical systems—systems to be used as rods and

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and that as a result the instruments do not detect motion through the aether—that is a perfectly coherent point of view" (ibid.). I agree with Bell that the theory he outlines is perfectly coherent and feasible. Yet, as I will show in chapter four, there are very strong arguments (of the potent 'common cause'-variety) for making this Dublin FitzGerald-Larmor-Bell interpretation of special relativity an extremely unattractive basis for an alternative to the standard Copenhagen interpretation of quantum mechanics.

[64] My views on this issue were shaped in discussions with Jon Dorling and Dennis Dieks.

[65] See, e.g., Sklar 1974, Friedman 1983, Earman 1989, and Norton 1992b.

[66] I am grateful to John Norton for pointing this out to me.

clocks, say—to probe the space-time structure. But we can still identify the properties we discover with these clocks and rods as properties of the space-time structure. By analogy, the position of a star is a purely astronomical fact that does not become at least in part optical because we happen to ascertain this fact with a telescope. So, on closer examination, the definition of kinematical licensed by modern philosophers of space and time is such that length contraction is purely kinematical.

I can understand why people with otherwise very different views, such as Dennis Dieks and John Stachel, are not fully satisfied with this analysis. It seems to reify the space-time structure more than necessary.[67] Length contraction, on the modern view, is a geometrical property of the space-time arena and our epistemic access to such properties is through the spatio-temporal behavior of systems in this arena. Why not simply identify length contraction with the spatio-temporal behavior of physical systems and avoid this distinction between a space-time arena and its contents? I sympathize with this position, and I will continue to cast my arguments in terms of the spatio-temporal behavior of physical systems. I will use the terms kinematics and dynamics to distinguish between the spatio-temporal behavior and other behavior of physical systems. However, I do want to emphasize that this way of distinguishing between kinematics and dynamics is perfectly in line with the distinction between kinematics and dynamics in the dominant tradition in modern philosophy of space and time.

Given this clarification of the notions kinematical and dynamical, is there still room for maintaining that length contraction is at least partially a dynamical effect? I think not. But I do want to address a very seductive neo-Lorentzian prejudice that might lead people to think otherwise. The problem is that the spatio-temporal behavior predicted by the special theory of relativity is highly counter-intuitive. It therefore seems to cry out for further explanation. This demand for further explanation, however, has no other basis than the completely unwarranted assumption that somehow the natural spatio-temporal behavior of systems is behavior in accordance with our every day Newtonian intuitions about space and time and that explanations are therefore required whenever we have deviations from that Newtonian norm.[68] The spatio-temporal behavior of systems in some possible world obviously depends on the space-time structure in that world. In the absence of gravitational fields, the spatio-temporal behavior of systems in our actual world is that of systems in a Minkowski space-time. I want to emphasize that in explaining this behavior nothing needs to be said over and above the simple assertion that

[67] Or, as John Stachel has put it, in Marxist fashion, it reeks of "fetishism of mathematics" (Stachel 1994, pp. 148–151).

[68] Cf. the analysis of explanation in van Fraassen 1980, pp. 134–153.

the structure of space-time in our actual world is that of a Minkowski space-time.[69] It simply makes no sense to demand an explanation for why the spatio-temporal behavior is not the spatio-temporal behavior one would have in some other space-time structure we can conjure up, unless, for some reason, one believes that, despite appearances to the contrary, the real structure of our actual space-time is not (not even to a good approximation) that of a Minkowski space-time, but, say, that of a Newtonian space-time, i.e., unless one has been unable to fully free oneself from neo-Lorentzian prejudices.

The upshot then is that, given my definition of the term ‘kinematical,’ one can only deny the second premise of my modus ponens argument if one assumes that, even in the absence of gravitational fields, the space-time structure of our actual world is not really Minkowskian. That, of course, provides very strong justification for that premise. So, what can be inferred from the insight that the Laue effect has the same status as length contraction, is that the Laue effect is purely kinematical, in the sense in which I use this term, not that length contraction is at least partially dynamical after all.

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[69] Notice that this claim does not hinge on the ontological status—substantivalist or relationalist—one wants to ascribe to the space-time structure.

2.4 A 'forces'-account of the Trouton-Noble experiment; the role of the relativity of simultaneity

2.4.1 Laue's attempt to give a more intuitive account of the Trouton-Noble experiment.

In his work on relativity theory in 1911–1912, Laue would emphasize again and again that the Laue effect exhibited in the Trouton-Noble experiment and in the Lewis-Tolman bent lever thought experiment (see section 2.5) strongly supports relativistic mechanics over classical Newtonian mechanics. In a short paper entitled "Remarks on the law for levers (*Hebelgesetz*) in the theory of relativity," for instance, he writes:

> To sustain straightline uniform motion [in Newtonian mechanics] no turning couple is needed. This is different in relativity theory: for a body with elastic stresses [to sustain straightline uniform motion] a turning couple is generally required. (Laue 1912a, p. 163)

After going over the example of the Lewis-Tolman bent lever, Laue returns to the Trouton-Noble experiment in the last paragraph of the paper:

> The point of this experiment, as is well-known, is to detect the turning couple that a condenser in uniform and straightline motion experiences from its electromagnetic field according to the unanimous prediction of all electromagnetic theories. The experimental result is that a rotation does not occur. One cannot conclude from this that the aforementioned turning couple does not exist. The material parts of the condenser contain elastic stresses and are therefore in need of a turning couple to move in a straight line without rotation. The turning couple exerted by the field is just the turning couple needed for this purpose—in this sense, the Trouton-Noble experiment decides in favor of the dynamics of relativity[70] theory and against Newtonian mechanics. (Ibid., p. 164)[71]

As Laue points out in the introduction of the paper "On the theory of the experiment of Trouton and Noble" published later that year, the support relativity theory receives from the Trouton-Noble experiment in this manner is much harder to appreciate than the support it receives from the Michelson-Morley experiment (Laue 1912b, p. 168). Part of the problem is that both his and Lorentz's treatment of the Trouton-Noble experiment is based on the new and unfamiliar concept of electromagnetic momentum (ibid., p. 169). As we saw in sections 1.4 and 2.3, both Lorentz (1904b) and Laue (1911a) use the equation $\mathbf{T} = -\,\mathbf{v} \times \mathbf{P}$ to compute the turning couple on the condenser (see Eq. 1.38 and Eqs. 2.59–2.60). A more intuitive account of the experiment would be obtained if the turning couple were to be evaluated through an equation

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[70] Laue calls 'the dynamics of relativity' what I (following the usage in Planck 1908) called the '*general* dynamics' of the special theory of relativity.

[71] With the advantage of over eight decades of hindsight, one will recognize two inaccuracies in this passage. First, electrodynamics does not necessarily predict a turning couple on a moving condenser. Under the Rohrlich definition of four-momentum and angular momentum, it does not. Second, as I showed in section 2.3, the experiment illustrates a purely kinematical effect in relativity theory. I want to emphasize that the correction of these inaccuracies actually strengthens rather than weakens Laue's argument in this passage.

of the familiar form $\mathbf{T} = \mathbf{x} \times \mathbf{F}$, directly giving the moments of the various forces. This is what Laue set out to do in this new paper devoted exclusively to the Trouton-Noble experiment. The question he wants to answer is:

> What are the pairs of forces producing the turning couple and where on the condenser do they act? (Laue 1912b, p. 169)

In section 1.2, I gave a simplified version of the 'forces'-account of the Trouton-Noble experiment that Laue offers to answer this question. Even that simplified version, I am afraid, does not quite succeed in lending the Trouton-Noble experiment the sort of intuitiveness (*Anschaulichkeit*, ibid., p. 168) that is so characteristic of the Michelson-Morley experiment. Laue's own rigorous version does far worse on this score. Laue's analysis is based on an exact solution for the field of the condenser, a solution taken from Helmholtz (Laue 1912b, p. 171). Since what happens at the edges of the plates of the condenser turns out to be responsible for half the turning couple (Laue 1912b, p. 176; Pauli 1921, pp. 129–130), it looks as if there is no way around dealing with the edge effects in this manner. Fortunately, this is not the case. Exactly how the field drops off at the edges does not affect the turning couple at all. The simple behavior at the edges that I assumed in section 2.3, where the field abruptly drops from its constant value inside the condenser to its zero value outside, gives the exact same turning couple as the far more complicated actual behavior at the edges. Rather than going over Laue's unnecessarily complicated derivation based on the solution taken from Helmholtz, I will therefore present a much simpler but essentially equivalent 'forces'-account of the Trouton-Noble experiment on the basis of the idealized solution for the field of the condenser in Eq. 2.87. My 'forces'-account will still not put the Trouton-Noble experiment on a par with the Michelson-Morley experiment in terms of intuitiveness, but it will give us a more intuitive grasp of some of the conclusions we reached in section 2.3

2.4.2 A stream-lined version of Laue's 'forces'-account of the Trouton-Noble experiment. As we saw in section 1.4, the starting point in Lorentz's derivation of the equation $\mathbf{T} = -\,\mathbf{v} \times \mathbf{P}$ both he and Laue used to compute the turning couple in the Trouton-Noble experiment was (see Eq. 1.43)

$$\mathbf{T} = \int \mathbf{x} \times \mathbf{f}\; d^3x, \tag{2.116}$$

where for $\mathbf{f}$ Lorentz substituted the Lorentz force density $\mathbf{f} = \rho\,(\mathbf{E} + \mathbf{v} \times \mathbf{B})$ (see Eq. 1.44). As I already pointed out in section 1.4 (see Eq. 1.54), this force density forms the spatial part of the relativistic four-force density f^μ, defined as (see, e.g., Laue 1911a, p. 139, Eq. (1)):

$$f^{\mu} \equiv -\partial_{\nu} T^{\mu\nu}. \tag{2.117}$$

From the expressions in Eq. 2.88 and Eq. 2.96 for $T'^{\mu\nu}_{\text{EM}}$ and $T'^{\mu\nu}_{\text{non-EM}}$ in the condenser of the Trouton-Noble experiment, it is clear that the four-force density vanishes everywhere except on the six sides of the condenser (cf. Eq. 2.90). The contributions to the turning couple will come from the force density on four of these six sides: the top and bottom plates and the front and rear ends of the condenser.

The forces on these four sides can all be represented by forces acting at their center without changing the value of the resulting turning couple. Consider, e.g., the force on the top plate. The uniform force density on this plate can be represented by a sharply peaked force density which is just the uniform density multiplied by (a) the area $A = ab$ of the top plate and (b) a delta function which is zero everywhere except on the worldline W_T of the center of the top plate. The force densities on the bottom plate and on the front and rear ends of the condenser can likewise be represented by force densities that are sharply peaked on the worldlines W_B, W_F, and W_R of the centers of these three sides. Fig. 2.8 shows the projection of these worldlines on the *xt*-plane of the x^{μ}-frame in which the condenser is moving with its plates tilted at angle θ with respect its velocity (cf. Fig. 2.5 and Fig. 2.6). The shaded area represents the bundle of worldlines W_C of the condenser as a whole.

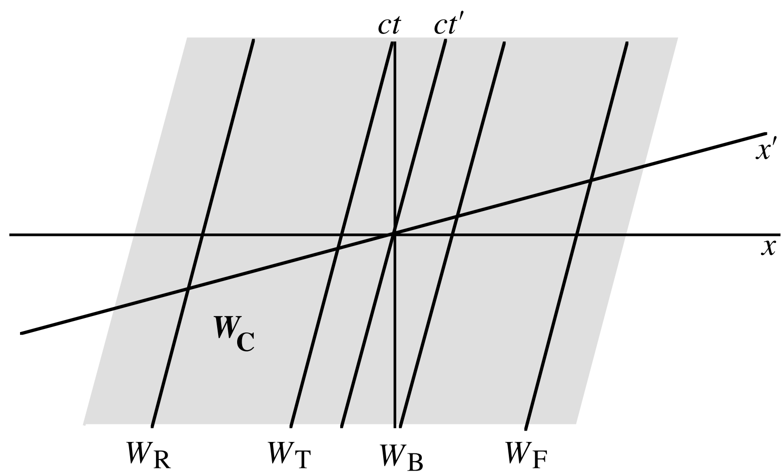

Figure 2.8 The worldlines of selected points of the condenser

For these sharply peaked force densities, Eq. 2.116 reduces to:

$$\mathbf{T} = \mathbf{x}_R \times \mathbf{F}_R + \mathbf{x}_T \times \mathbf{F}_T + \mathbf{x}_B \times \mathbf{F}_B + \mathbf{x}_F \times \mathbf{F}_F \tag{2.118}$$

where $\mathbf{F}_R$ and $\mathbf{F}_F$ are equal to db times the constant force density $\mathbf{f}$ on the rear and front ends of the condenser, respectively (db is the area of these sides); and where $\mathbf{F}_T$ and $\mathbf{F}_B$ are equal to ab

times the constant force density **f** on the top and bottom plates, respectively (ab is the area of the plates). Eq. 2.118 has just the form Laue was after in his ‘forces’-account of the Trouton-Noble experiment. It shows the forces responsible for the turning couple and where they act.

We can write down Eq. 2.118 both for the electromagnetic and for the non-electromagnetic forces. The non-electromagnetic forces will, of course, just be the opposite of the electromagnetic forces, so that there is no net turning couple on the condenser.

The easiest way to calculate the various forces in Eq. 2.118 is as follows. First, I calculate the forces in the x'^{μ}-frame I used in section 2.3 in which the plates are perpendicular to the y-axis and in which both the electromagnetic and the non-electromagnetic energy-momentum tensors are diagonal. Then, I calculate the forces in a new rest frame, that I will continue to call the x'^{μ}-frame, in which the plates are tilted at an angle θ' with respect to the xz-plane. Finally, I use Planck’s transformation law for forces (see Eq. 1.9) to find the forces in the x^{μ}-frame in which the condenser is moving in the x-direction with its plates tilted at an angle θ with respect to its velocity **v**. This final step is completely analogous to the calculation I went through in the simplified ‘forces’-account in section 1.2 (Eqs. 2.5–2.15). I will do the calculations for the electromagnetic forces. The non-electromagnetic forces only differ from these by a minus sign.

To compute the electromagnetic force density on the various sides of the condenser in the conveniently chosen rest frame of section 2.3, I evaluate the spatial components of Eq. 2.117 for the condenser’s electromagnetic energy-momentum tensor $T'^{\mu\nu}_{\mathrm{EM}}$ (see Eq. 2.88):

$$f'^{i}_{\mathrm{EM}} = -\partial'_{\nu} T'^{i\nu}_{\mathrm{EM}}. \tag{2.119}$$

Consider the x-component of this equation:

$$f'^{\mathrm{EM}}_{x} = -\,\partial'_{1} T'^{11}_{\mathrm{EM}} = -\,u' \,\frac{\partial f(x',y',z')}{\partial x'}. \tag{2.120}$$

Inserting Eq. 2.89 for $f(x',y',z')$, we find

$$f'^{\mathrm{EM}}_{x} = -\,u' \left[\delta\!\left(\frac{a}{2}+x'\right) - \delta\!\left(\frac{a}{2}-x'\right)\right] \theta\!\left(\frac{d}{2}-|y'|\right) \theta\!\left(\frac{b}{2}-|z'|\right), \tag{2.121}$$

where I used that $\theta\!\left(\frac{a}{2}-|x'|\right)$ can be written as

$$\theta\!\left(\frac{a}{2}-|x'|\right) = \theta\!\left(\frac{a}{2}+x'\right) \theta\!\left(\frac{a}{2}-x'\right), \tag{2.122}$$

and that the derivative of the step-function $\theta(x)$ is the Dirac delta function $\delta(x)$. So, we have a uniform force density $f'^{\mathrm{EM}}_{x} = -u'$ on the $y'z'$-side of the condenser at $x' = -a/2$ (the rear end); a uniform force density $f'^{EM}_x = u'$ on the $y'z'$-side of the condenser at $x' = a/2$ (the front end); and $f'^{EM}_x = 0$ everywhere else. So, in the x'^{μ}-frame, we have forces $\pm\mathbf{F}'^{//}_{EM}$ at the center of the front and rear ends of the condenser, which are parallel to the plates (hence the superscript '//') and which are of magnitude

$$F'^{//}_{EM} = u' \cdot db = \frac{U'_{EM}}{a}, \tag{2.123}$$

where I used that $u' = U'_{EM}/V'$ (see Eq. 2.98) and that $V' = abd$.

A completely analogous argument can be given for the y- and z-components of $\mathbf{f}'_{EM}$. Since the forces in the z-direction do not contribute to the turning couple, I will not bother to write down the result for f'^{EM}_z. The result for f'^{EM}_y is as follows. There is a uniform force density $f'^{EM}_y = u'$ on the $x'z'$-side of the condenser at $y' = -d/2$ (the bottom plate); a uniform force density $f'^{EM}_y = -u'$ on the $x'z'$-side of the condenser at $y' = d/2$ (the top plate); and $f'^{EM}_y = 0$ everywhere else. So, in the x'^{μ}-frame, we have forces $\pm\mathbf{F}'^{\perp}_{EM}$ at the center of the top and bottom plates of the condenser, which are perpendicular to the plates (hence the superscript '⊥') and which are of magnitude

$$F'^{\perp}_{EM} = u' \cdot ab = \frac{U'_{EM}}{d}. \tag{2.124}$$

In Fig. 2.9, these forces, along with the forces $\mathbf{F}'^{//}_{non\text{-}EM}$ and $\mathbf{F}'^{\perp}_{non\text{-}EM}$ exactly opposite to them, are drawn in a rest frame in which the plates of the condenser are tilted with respect to the xz-plane at an angle θ'. In the remainder of the section, I will call this rest frame the x'^{μ}-frame.

Fig. 2.9 is misleading in that the $\mathbf{F}'^{\perp}$ forces are actually much bigger than the $\mathbf{F}'^{//}$ forces. From Eqs. 2.123–2.124, it follows that

$$\frac{F'^{\perp}}{F'^{//}} = \frac{a}{d} \ . \tag{2.125}$$

Since, in order to be able to ignore edge effects, we had to assume that the length a of the plates is much larger than the distance d between them, it will be the case that $a/d \gg 1$. At first glance, this suggests that we were fully justified in section 1.2 to focus on the $\mathbf{F}'^{\perp}$ forces at the center of the plates and to ignore the $\mathbf{F}'^{//}$ forces at the edges. However, the turning couples coming from the small $\mathbf{F}'^{//}$ forces (in a frame in which the condenser is moving) will be of the same order of magnitude as the turning couples coming from the $\mathbf{F}'^{\perp}$ forces, because the small forces at the edges have a big arm of the order of a, whereas the big forces at the center have a small arm of the order of d. Actually, as we will see shortly, the two turning couples are exactly equal (cf. Laue 1912b, p. 176; Pauli 1921, pp. 129–130).

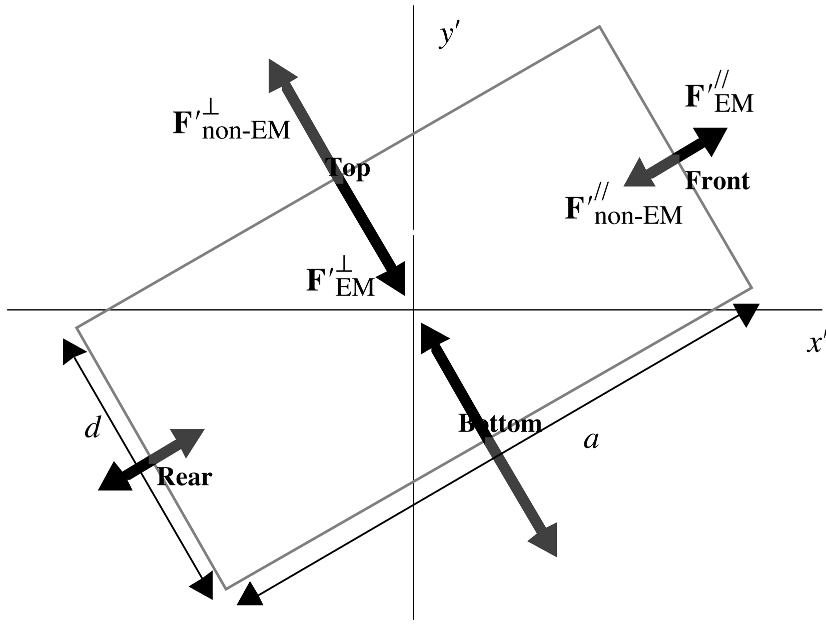

Figure 2.9 Forces in a charged condenser at rest.

Now that we know the forces on the condenser at rest, we can apply the same argument we applied in section 1.2 to find the turning couples produced by the various forces on the condenser in motion. I will go through the calculation for the total turning couple produced by the electromagnetic forces. The total turning couple produced by the non-electromagnetic forces will be equal and opposite to that of the electromagnetic turning couple.

For a condenser at rest, the electromagnetic forces, of course, do not give a turning couple at all:

$$\mathbf{T}'_{\mathrm{EM}} = \mathbf{d}' \times \mathbf{F}'^{\perp}_{\mathrm{EM}} + \mathbf{a}' \times \mathbf{F}'^{//}_{\mathrm{EM}} = 0, \tag{2.126}$$

where $\mathbf{d}'$ is the vector from the point labeled ‘bottom’ to the point labeled ‘top’ in Fig. 2.9 and $\mathbf{a}'$ is the vector from ‘rear’ to ‘front’. $\mathbf{F}'^{\perp}_{\mathrm{EM}}$ and $\mathbf{F}'^{//}_{\mathrm{EM}}$ are the electromagnetic forces at ‘top’ and ‘front’, respectively.

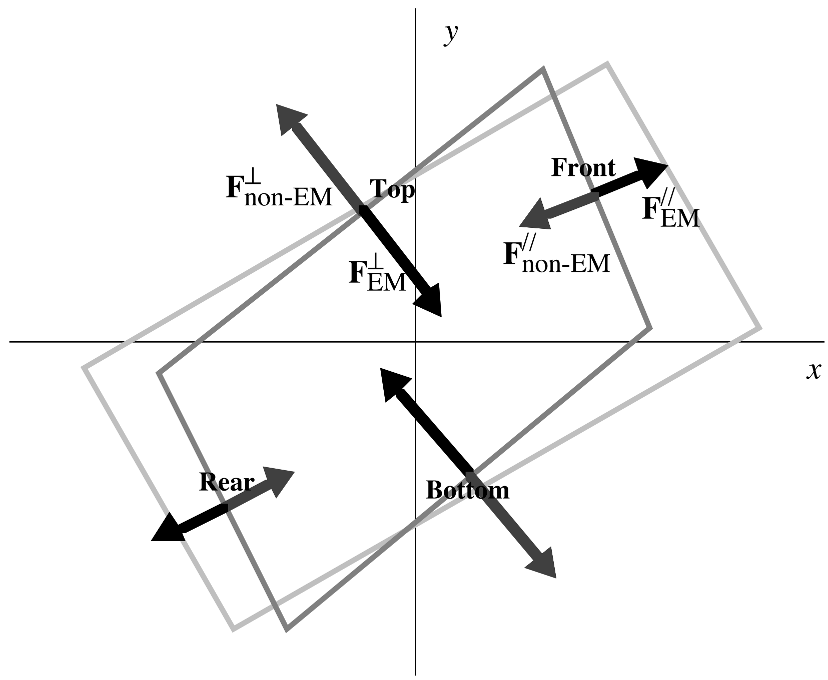

Figure 2.10 Forces in a charged moving condenser.

Fig. 2.10 shows the corresponding forces on the condenser in motion. As will be shown below, $\mathbf{F}^{\perp} \perp \mathbf{a}$ and $\mathbf{F}^{//} \perp \mathbf{d}$.[72] These forces do give a non-vanishing turning couple

$$\mathbf{T}_{\text{EM}} = \mathbf{d} \times \mathbf{F}^{\perp}_{\text{EM}} + \mathbf{a} \times \mathbf{F}^{//}_{\text{EM}}. \tag{2.127}$$

From the geometry of Figs. 2.9 and 2.10, one finds for **d** and **a**:

$$\begin{aligned} \mathbf{d} &= \text{diag}(1/\gamma, 1, 1)\ \ \mathbf{d}' = d\left(-\frac{\sin\theta'}{\gamma}, \cos\ \theta', 0\right) \\ \mathbf{a} &= \text{diag}(1/\gamma, 1, 1)\ \ \mathbf{a}' = a\left(\frac{\cos\theta'}{\gamma}, \sin\ \theta', 0\right). \end{aligned} \tag{2.128}$$

Using Planck's transformation law for forces (Eq. 1.9), one finds for $\mathbf{F}^{\perp}_{\text{EM}}$ and $\mathbf{F}^{//}_{\text{EM}}$:

$$\begin{aligned} \mathbf{F}^{\perp}_{\text{EM}} &= \text{diag}(1, 1/\gamma, 1/\gamma)\ \mathbf{F}'^{\perp}_{\text{EM}} = \frac{U'_{\text{EM}}}{d}\left(\sin\theta', -\frac{\cos\theta'}{\gamma}, 0\right) \\ \mathbf{F}^{//}_{\text{EM}} &= \text{diag}(1, 1/\gamma, 1/\gamma)\ \mathbf{F}'^{//}_{\text{EM}} = \frac{U'_{\text{EM}}}{a}\left(\cos\theta', \frac{\sin\theta'}{\gamma}, 0\right). \end{aligned} \tag{2.129}$$

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[72] As Lorentz already noticed in his treatment of electrostatics in frames of reference moving uniformly with respect to the ether, a Coulomb force in a system at rest in the ether which is perpendicular to a certain surface becomes a Coulomb force perpendicular to the corresponding surface in the corresponding state moving with respect to the ether (see, e.g., Lorentz 1899b, p. 261).

Inserting Eqs. 2.128–2.129 into Eq. 2.127, we arrive at:[73]

$$\begin{aligned}\mathbf{T}_{\rm EM} &= \left(0, 0, \left[d_x F^{\perp}_{{\rm EM}_y} - d_y F^{\perp}_{{\rm EM}_x}\right] + \left[a_x F^{//}_{{\rm EM}_y} - a_y F^{//}_{{\rm EM}_x}\right]\right) \\ &= \left(0, 0, 2\ U'_{\rm EM}\left[\frac{\sin\theta' \cos\theta'}{\gamma^2} - \cos\theta' \sin\theta'\right]\right) \\ &= \left(0, 0, -\ U'_{\rm EM}\,\beta^2 \sin 2\theta'\right).\end{aligned} \tag{2.130}$$

which agrees with the results found in chapter one (see, e.g., Eq. 1.39[74]).

2.4.3 Forces, energy, and the relativity of simultaneity: Einstein on situations similar to those in the Trouton-Noble experiment. The expressions for the forces on the condenser in Eq. 2.129 can be used to gain some insight into the odd transformation behavior of the condenser's electromagnetic and non-electromagnetic energy in the Laue picture. In the Rohrlich picture, the total energy in the x^μ-frame, $U_{\rm tot} = \gamma U'_{\rm tot}$ (see Eq. 2.99), is divided into (see Eq. 2.114)

$$U^{\rm R}_{\rm EM} = \gamma U'_{\rm EM}, \qquad U^{\rm R}_{\text{non-EM}} = \gamma U'_{\text{non-EM}}. \tag{2.131}$$

In the Laue picture, the total energy in the x^μ-frame is divided into (see Eqs. 2.104–2.105):[75]

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[73] From Eqs. 2.128–2.129, it also follows that $\mathbf{F}^{\perp} \cdot \mathbf{a} = 0$ and $\mathbf{F}^{//} \cdot \mathbf{d} = 0$. Hence, Fig. 2.10 accurately reflects that $\mathbf{F}^{\perp}$ is perpendicular to **a**, and that $\mathbf{F}^{//}$ is perpendicular to **d**.

[74] Notice that to order β^2 we can set $\theta' = \theta$.

[75] With the help of Eqs. 2.131 and 2.132, we can assess Larmor's 'energy'-account of the Trouton-Noble experiment (see section 1.3) from the point of view of both the Laue and the Rohrlich account of the experiment in special relativity.

From the Laue point of view, the assessment of Larmor's analysis is the same as from the point of view of Lorentz's theory (see section 1.4, Eqs. 1.40–1.42). Contrary to what Larmor claimed, the electromagnetic energy in a moving condenser does depend on θ, the condenser's orientation with respect to its velocity even if we assume the Lorentz-FitzGerald contraction. It does not follow, however, that there is a net turning couple. The non-electromagnetic energy—and this would probably have come as somewhat of a surprise to Larmor who tacitly and quite reasonably assumed that the non-electromagnetic part of the condenser would satisfy the Galilean relativity principle—also depends on θ, in such a way that the total energy does not.

The Rohrlich picture of the Trouton-Noble experiment is considerably closer to Larmor's account of the experiment. In the Rohrlich picture, as in the Larmor picture, there is no electromagnetic turning couple on a moving condenser. In both cases, this can be understood on the basis of the θ-independence of the electromagnetic energy of the condenser. However, even from the Rohrlich point of view, Larmor's analysis can not be accepted. No matter whether we consider Larmor's derivation of the electromagnetic energy in a moving condenser from the Laue point of view or from the Rohrlich point of view, Larmor fails to take into account the energy necessary to build up the momentum that is generated upon charging the moving condenser. This energy is equal to $v P_x$ (see Eq. 1.40). Using Eq. 2.115 for P_x, we find that this energy is equal to $\gamma\, \beta^2 U'_{\rm EM}$ in the Rohrlich picture. Adding this to the energy $U'_{\rm EM}/\gamma$ found by Larmor (see Eq. 1.20), we recover Eq. 2.131 in the Rohrlich picture:

$$U^{L}_{EM} = \gamma U'_{EM} + \gamma U'_{EM}\beta^2\cos^2\theta' - \gamma U'_{EM}\beta^2\sin^2\theta',$$
$$U^{L}_{non\text{-}EM} = \gamma U'_{non\text{-}EM} - \gamma U'_{EM}\beta^2\cos^2\theta' + \gamma U'_{EM}\beta^2\sin^2\theta'. \qquad (2.132)$$

With the help of Eq. 2.129, we can interpret the puzzling θ'-dependent terms in Eq. 2.132 directly in terms of the relativity of simultaneity. This very elegant way of looking upon energy transformation equations such as Eq. 2.132 is due to Einstein (1907b, pp. 373–377).[76]

Look back at the Minkowski diagram in Fig. 2.8 showing the worldlines of the points of the condenser at which the forces act, along with two frames of reference, the x'^{μ}-frame in which the condenser is at rest, and the x^{μ}-frame in which it is moving. At some point, viz. when the condenser was charged, the forces on the condenser were "switched on." From the way we proceeded—calculating the field in the x'^{μ}-frame and then transforming to the x^{μ}-frame—it is clear that we have been considering a situation in which the condenser was charged in its rest frame. This means that the forces were switched on simultaneously in the x'^{μ}-frame. Hence, they were not switched on simultaneously in the x^{μ}-frame. In the x^{μ}-frame, as can be read off directly from the Minkowski diagram in Fig. 2.8, the forces at the rear end of the condenser were switched on first, followed by the ones at the top, the ones at the bottom, and finally the ones at the front end of the condenser. This insight is the key to understanding the θ'-dependent terms in Eq. 2.132.

Suppose the forces were switched on in the x'^{μ}-frame at $t'=0$. Let 'R_0,' 'T_0,' 'B_0,' and 'F_0' be the points in space-time at which the forces $\mathbf{F}_R$, $\mathbf{F}_T$, $\mathbf{F}_B$, and $\mathbf{F}_F$, respectively, were switched on. As one can see directly in Fig. 2.8, this means that R_0, T_0, B_0, and F_0 are the intersections of the x'-axis with the worldlines W_R, W_T, W_B, and W_F, respectively. The x^{μ}-coordinates of these events are related to their x'^{μ}-coordinates through $x^{\mu} = \Lambda^{\mu}{}_{\nu}\, x'^{\nu}$, where $\Lambda^{\mu}{}_{\nu}$ is the matrix for a

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$$U'_{EM}/\gamma + \gamma\beta^2 U'_{EM} = \gamma U'_{EM}\left(\frac{1}{\gamma^2} + \beta^2\right) = \gamma U'_{EM}.$$

The discrepancy between the Larmor and the Rohrlich accounts of the Trouton-Noble experiment can also, and perhaps more intuitively, be expressed as follows. Larmor, in effect, calculated the energy of the condenser in a co-moving Galilean frame (see sections 1.2 and 1.3). An energy U'_{EM}/γ for a Galilean co-moving observer is an energy U'_{EM} for a Lorentzian co-moving observer (cf. Fig. 2.5: the Galilean co-moving observer multiplies the energy density in the condenser's rest frame by the length of the intersection $\Sigma_L \cap W_C$; the Lorentzian co-moving observer multiplies that same energy density by the length of the intersection $\Sigma_R \cap W_C$). For an observer at rest in the ether this will be an energy $\gamma U'_{EM}$, the sum of the rest energy U'_{EM} and the kinetic energy $U'_{EM}(1-\gamma)$. This is just the energy in Eq. 2.131. Lacking the notion of the inertia of energy, Larmor did not take into account the kinetic energy term.

I also want to emphasize that it does not make much sense to credit Larmor in 1902 with the Rohrlich idea of integrating electromagnetic energy density over a hyperplane of simultaneity in the condenser's rest frame.

[76] Cf. Pauli 1921, pp. 126–127. See also Becker 1962, I, pp. 400–401 and Norton 1992a, pp. 48–49. I am grateful to Michael Pointer for drawing my attention to this argument of Einstein.

boost in the negative x-direction. Consider the force $\mathbf{F}^{\perp}_{\text{EM}}$ on the top plate and the force $-\mathbf{F}^{\perp}_{\text{EM}}$ on the bottom plate. The former is switched on a time interval $t_{\text{B}_0} - t_{\text{T}_0}$ before the latter. Using $x^{\mu} = \Lambda^{\mu}{}_{\nu}\, x'^{\nu}$, we can calculate the length of this time interval:

$$c\,(t_{\text{B}_0} - t_{\text{T}_0}) = \gamma\left(ct'_{\text{B}_0} + \beta x'_{\text{B}_0}\right) - \gamma\left(ct'_{\text{T}_0} + \beta x'_{\text{T}_0}\right). \tag{2.133}$$

Inserting $t'_{\text{B}_0} = t'_{\text{T}_0}$ and $x'_{\text{B}_0} - x'_{\text{T}_0} = d \sin\theta'$, we obtain:

$$t_{\text{B}_0} - t_{\text{T}_0} = \frac{1}{c}\,\gamma\beta\, d \sin\theta' \tag{2.134}$$

During this time interval, the condenser moves a distance $v(t_{\text{B}_0} - t_{\text{T}_0})$ and the force $\mathbf{F}^{\perp}_{\text{EM}}$ on the top plate does a positive amount of work ΔW_1 that is not compensated by a negative amount of work done by the force $-\mathbf{F}^{\perp}_{\text{EM}}$ on the bottom plate, which has not been switched on yet. Using Eq. 2.129 for $\mathbf{F}^{\perp}_{\text{EM}}$, we find that:

$$\begin{aligned} \Delta W_1 &= \mathbf{F}^{\perp}_{\text{EM}_x}\, v\,(t_{\text{B}_0} - t_{\text{T}_0}) \\ &= \frac{U'_{\text{EM}}}{d} \sin\theta' \, \frac{v}{c}\, \gamma\, \beta d \sin\theta' \\ &= U'_{\text{EM}}\, \gamma \beta^2 \sin^2\theta' . \end{aligned} \tag{2.135}$$

The change ΔU_1 in energy of the system is minus the work ΔW_1 done by the system. Hence, the fact that $\mathbf{F}^{\perp}_{\text{EM}}$ on the top plate is switched on before $-\mathbf{F}^{\perp}_{\text{EM}}$ on the bottom plate is responsible for a contribution

$$\Delta U_1 = -\Delta W_1 = -U'_{\text{EM}}\, \gamma \beta^2 \sin^2\theta' . \tag{2.136}$$

to the total energy of the system. This is precisely the last term in the expression for U^{L}_{EM} in Eq. 2.132. The other three θ'-dependent terms in Eq. 2.132 can be interpreted in exactly the same way.

Consider the force $-\mathbf{F}^{//}_{\text{EM}}$ at the rear end of the condenser and the force $\mathbf{F}^{//}_{\text{EM}}$ at the front end. The former is switched on a time interval $t_{\text{F}_0} - t_{\text{R}_0}$ before the latter. The length of this time interval is given by (cf. Eqs. 2.133–2.134):

$$t_{\text{F}_0} - t_{\text{R}_0} = \frac{1}{c}\,\gamma\beta\, a \cos\theta' . \tag{2.137}$$

During this time interval, the condenser moves a distance $v(t_{\text{F}_0} - t_{\text{R}_0})$ and the force $-\mathbf{F}^{//}_{\text{EM}}$ at the rear end does a negative amount of work ΔW_2 that is not compensated by a positive amount of work done by the force $\mathbf{F}^{//}_{\text{EM}}$ at the front end, which has not been switched on yet. Using Eq. 2.129 for $\mathbf{F}^{//}_{\text{EM}}$, we find that:

$$\begin{aligned} \Delta W_2 &= -\mathbf{F}^{//}_{\text{EM}_x}\, v\,(t_{\text{F}_0} - t_{\text{R}_0}) \\ &= -\frac{U'_{\text{EM}}}{a}\cos\theta'\,\frac{v}{c}\,\gamma\,\beta a\cos\theta' \\ &= -U'_{\text{EM}}\,\gamma\beta^2\cos^2\theta'. \end{aligned} \tag{2.138}$$

This gives a contribution $\Delta U_2 = -\Delta W_2$ to the total energy of the system. This is just the second term in the expression for U^{L}_{EM} in Eq. 2.132. The non-electromagnetic forces $\mathbf{F}^{\perp}_{\text{non-EM}}$ and $\mathbf{F}^{//}_{\text{non-EM}}$ will likewise give contributions $-\Delta U_1$ and $-\Delta U_2$, which account for the θ'–dependent terms in the expression for $U^{\text{L}}_{\text{non-EM}}$ in Eq. 2.132.

This is a remarkable insight on Einstein's part. It amounts to realizing that the θ'–dependent terms in the expressions for the electromagnetic and non-electromagnetic energy in the x^μ-frame stem from the fact that we evaluate the system's energy over a hyperplane of simultaneity in the x^μ-frame rather than over a hyperplane of simultaneity in the x'^μ-frame, the system's rest frame. This, of course, is just the difference between the Laue and the Rohrlich picture that we analyzed in great detail in section 2.3. Recall that the hyperplane Σ_{L} used to define the system's energy in the Laue picture is a hyperplane of simultaneity in the x^μ-frame, whereas the hyperplane Σ_{R} used to define the system's energy in the Rohrlich picture is a hyperplane of simultaneity in the x'^μ-frame. We can thus credit Einstein in 1907 with a deeper understanding of this aspect of the Trouton-Noble experiment than Laue in 1911–1912.

2.4.4 Toward a more intuitive understanding of the kinematical nature of the Laue effect. As I mentioned in the introduction to chapter two, the relativity of simultaneity is also what makes it possible for a closed non-static system—as opposed to a closed static system—to rotate in one frame but not in another. Consider the situation in Fig. 2.11, which shows the plates of a charged condenser at rest in the x'^μ-frame suspended on wires that prevent the plates from collapsing onto one another under the influence of their mutual Coulomb attraction. The plates, the charges they carry, the field these charges generate, and the supporting wires form a closed static system. At the points marked '×' in Fig. 2.11, we have devices that can cut the wires at those points. These devices are activated by built-in timers.[77] Suppose that the timers are all synchronized in the x'^μ-frame and that they activate the cutting devices at $t' = 0$. At that point, the plates will start accelerating toward one another. Before $t' = 0$, the plates, the

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[77] I am grateful to John Norton for suggesting this model to me.

charges they carry, and the electromagnetic field form an open static system: the system exchanges energy-momentum with the wires. After $t' = 0$, the plates, the charges they carry, and the electromagnetic field form an closed non-static system. No energy-momentum is entering or leaving the system. After a short time, the plates will collide, neutralize each other's charge, and annihilate the electromagnetic field. Fig. 2.11 shows the plates at three consecutive moments labeled $t' = 0, 1, 2$ (needless to say, these numbers were chosen quite arbitrarily). The values for t' are written next to the positions of the rear and front ends of both the top and the bottom plate (indicated by 'T^R,' 'T^F,' 'B^R,' and 'B^F') at those instants.

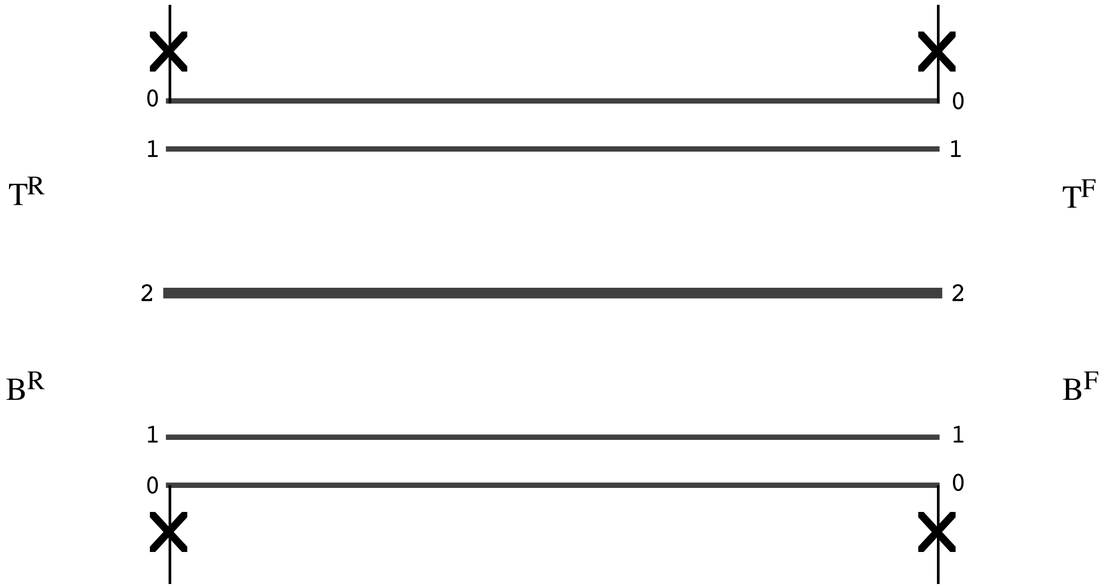

Figure 2.11 Condenser plates suspended on wires that are cut simultaneously in the condenser's rest frame.

In Fig 2.12, this same situation is shown in the x^μ-frame in which the condenser is moving at a velocity **v** in the x-direction. If the wires are cut simultaneously in the x'^μ-frame, they are not cut simultaneously in the x^μ-frame. The wires at the rear end will be cut before the wires at the front end (cf. Fig. 2.8). Suppose the wires at the rear are cut at $t = 0$ and the wires at the front are cut at $t = 1$. Further suppose that the clocks in the moving frame run slow by a factor .75 (which means that the condenser also will be contracted by a factor .75). In combination with Fig. 2.11, this information allows us to reconstruct what happens in the x^μ-frame. Fig. 2.12 shows the positions of the two plates at t = 0, .75, 1.5, 2.25.

Comparing Fig. 2.12 to Fig. 2.11, one sees that the plates rotate in the x^μ-frame (in the way indicated by the arrows in Fig. 2.12), whereas they do not rotate in the x'^μ-frame. This is just what one would expect on the basis of the Laue picture of what happens in the Trouton-Noble experiment. If one of the delicately balanced turning couples in the x^μ-frame were suddenly switched off, the remaining turning couple would cause the plates to start rotating. In the x'^μ-frame, however, there are no turning couples, so there will be no rotation.

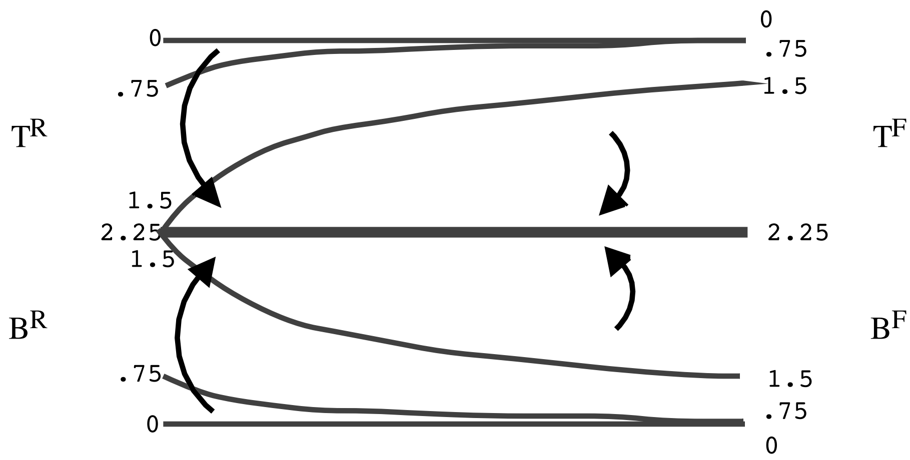

Figure 2.12 Condenser plates suspended on wires that are cut
at different times in a frame in which the condenser is moving.

I want to emphasize that cutting the wires in the setup we considered in Figs. 2.11–2.12 is not equivalent to switching off the turning couple coming from the non-electromagnetic part of the system. In terms of the forces shown in Fig. 2.10, cutting the wires means switching off the forces $\pm\mathbf{F}^{\perp}_{\text{non-EM}}$ that keeps the plates apart. We still have the forces $\pm\mathbf{F}^{//}_{\text{non-EM}}$, however, that keep the charges at the edges of the plates in place. Figs. 2.11–2.12 are meant to illustrate that there is no contradiction in the plates starting to rotate in the x^{μ}-frame but not in the x'^{μ}-frame; they are not meant to give an accurate picture of what would happen to the system if we were to annihilate the entire non-electromagnetic part of the system in one instant in the x'^{μ}-frame.

It will nonetheless be instructive to reflect some more upon the situation in Figs. 2.11–2.12, for it may help us understand how something sounding as dynamical as a turning couple can actually be a purely kinematical effect. As is clear from the construction of Fig. 2.12 out of Fig. 2.11, the rotation in the x^{μ}-frame is a purely kinematical effect. It is a direct manifestation of the relativity of simultaneity. That does not mean, however, that one cannot point to forces in the x^{μ}-frame that would seem to cause that rotation. Does this not mean that we are dealing with a dynamical effect after all?

At this point, it will be helpful to invoke an analogy due to Jon Dorling.[78] Consider a piece of chalk in ordinary three-dimensional Euclidean space. Hold the piece of chalk in front of you, in such a way that it lies in the plane perpendicular to your line of vision. Now rotate the piece of chalk out of this plane. Clearly, as you rotate the piece of chalk, the length of its projection onto the plane will decrease. This is part of the ordinary spatial behavior of objects in Euclidean space. Hence, it is a kinematical effect. However, if one considers the components of the forces

[78] Private communication. It is fair to say, I think, that the analogy by itself mainly preaches to the converted. In combination with the arguments I offered in section 2.3, however, it will serve the purpose, I hope, of getting a more intuitive grasp of how it can be that such things as turning couples can be kinematical effects.

holding the piece of chalk together, one notices that they also change as the piece of chalk is rotated. It is, in fact, a highly non-trivial fact about our actual world that the forces holding the piece of chalk together are such that the spatial behavior of the piece of chalk is the Euclidean spatial behavior we observe. Still, as I argued at the end of section 2.3, this does not mean that the phenomenon we observe in rotating the piece of chalk is dynamical. No explanation of this phenomenon is required over and above the assertion that the structure of space in our actual world is (to a very good approximation) Euclidean.

The same type of argument applies to the spatio-temporal behavior of systems in Minkowski space-time. Dorling likes to compare the spatial behavior of a piece of chalk that is rotated in Euclidean space to the spatio-temporal behavior of a rod that is set in motion in Minkowski space-time. The length of the projection onto the plane perpendicular to the observer's line of vision in the case of the piece of chalk then gets replaced by the standard definition of the length of a moving rod, i.e., the length of the intersection of the rod's bundle of worldlines with a spacelike hyperplane perpendicular to the observer's worldline. Just as the projection of the piece of chalk contracts as the piece of chalk is rotated, the length of the rod under this standard definition contracts. With both contractions there are corresponding changes in the components of the forces holding the rod or the piece of chalk together. And it is a highly non-trivial fact about our actual world that the forces holding the rod together are such that the spatio-temporal behavior of the rod is the Minkowskian spatio-temporal behavior we observe. Still, this does not make length contraction a dynamical effect. No explanation of this phenomenon is required over and above the assertion that the structure of space-time in our actual world is (to a very good approximation) Minkowskian.[79]

The Laue effect provides a striking illustration of Dorling's point. It is a highly non-trivial fact about our actual world that the forces on a charged condenser are such that, in the (standard) Laue picture of what happens in the experiment shown in Figs. 2.11–2.12, one gets a rotation in one frame but not in another. Still, this highly counter-intuitive behavior is just part of the normal Minkowskian spatio-temporal behavior and requires no explanation over and above the assertion that the structure of space-time is Minkowskian. The Laue effect, in other words, is purely kinematical.

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79 For a full and elegant treatment of the geometry of Minkowski space-time as a natural extension of Euclidean geometry from ordinary 3-dimensional space to 3+1-dimensional space-time, see Dorling 1982.

2.5 The experiments of Trouton and Noble and two well-known thought experiments in special relativity

2.5.1 The Lewis-Tolman bent lever and the Trouton-Noble experiment. Shortly after Laue published his first expositions of the Laue effect (Laue 1911a, 1911b), Sommerfeld drew his attention to a puzzling thought experiment in a paper by Lewis and Tolman (1909), in which the authors tried to develop relativistic mechanics independently of electrodynamics.[80] Laue must have been overjoyed to discover that this thought experiment ceases to be puzzling the moment one recognizes the role of the Laue effect in it. Laue wasted no time and published a short paper on this new manifestation of the Laue effect, emphasizing the close analogy between Lewis and Tolman's thought experiment and the Trouton-Noble experiment (Laue 1911c).[81] Early in 1912, he published another short paper on these findings (Laue 1912a). Laue would also include the thought experiment of Lewis and Tolman in later editions of his 1911 textbook on relativity (see, e.g., Laue 1952, pp. 164–165).

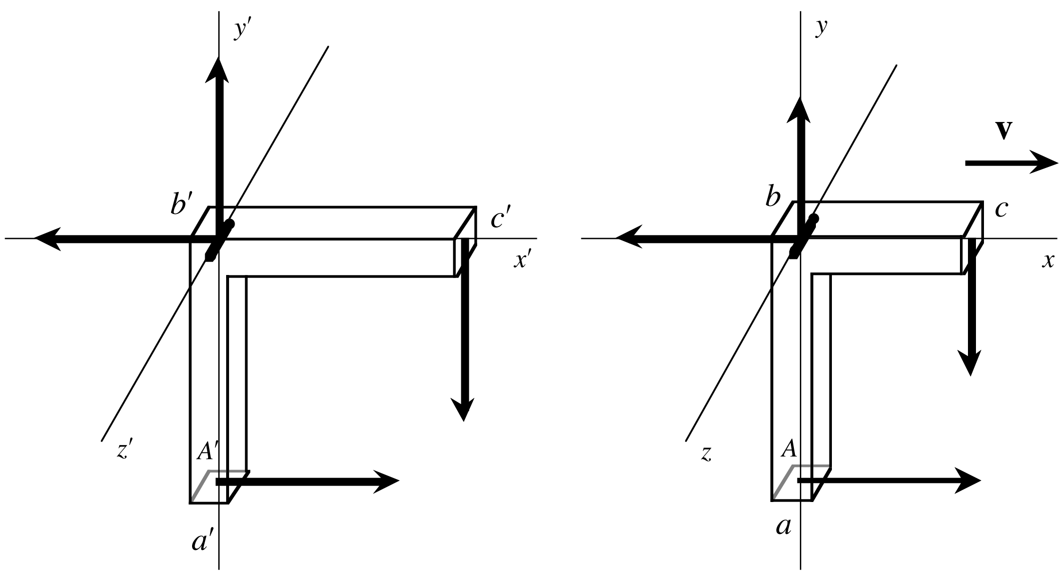

Figure 2.13 The Lewis-Tolman bent lever.

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[80] My source of information for Sommerfeld's role is a footnote in Laue 1911c, p. 514. Laue and Sommerfeld were colleagues in Munich at the time.

[81] The submission date for Laue 1911a is April 30, 1911; the submission date for Laue 1911c is July 5, 1911.

The analysis of the Trouton-Noble experiment in sections 2.3 and 2.4, especially the 'forces'-account of the experiment given in section 2.4, provides us with all the tools necessary to give Laue's analysis of the thought experiment of Lewis and Tolman.[82]

Fig. 2.13 shows the system considered by Lewis and Tolman, a system that has come to be known as 'the right-angled lever'[83] or as 'Lewis and Tolman's bent lever.'[84] On the left, it is shown in the x'^{μ}-frame, in which it is at rest; on the right, it is shown in the x^{μ}-frame, in which it is moving at a velocity **v** in the direction of the positive x-axis (cf. Figs. 2.9 and 2.10 for the analogous case of the Trouton-Noble condenser). Consider the system in its rest frame. It consist of two arms of equal length l', one ($a'b'$) along the y'-axis and one ($b'c'$) along the x$'$-axis. There is a pivot at b', restricting the system's freedom to move to rotations around the z'-axis. Two forces of equal magnitude F' are applied to the system, one at c' in the direction of the negative y'-axis and one at a' in the direction of the positive x'-axis. There will be reaction forces of the same size but in the opposite direction at b'. The system is in equilibrium. The net turning couple $\mathbf{T}'$ of the forces with respect to the z'-axis—the only axis around which the system can rotate—vanishes:

$$\begin{aligned} \mathbf{T}' &= \mathbf{x}'_{\mathrm{a}'} \times \mathbf{F}'_{\mathrm{a}'} + \mathbf{x}'_{\mathrm{c}'} \times \mathbf{F}'_{\mathrm{c}'} \\ &= (0, 0,\ F'l') + (0, 0, -F'l') = (0, 0, 0)\ . \end{aligned} \tag{2.139}$$

Now look at the drawing on the right of Fig. 2.13, showing the bent lever in the x^{μ}-frame. The arm ab still has the same length l', but the arm bc is contracted to l'/γ. Similarly, the forces in the x-direction are still of size F', but the forces in the y-direction are only of size F'/γ (see Eq. 1.9). As a consequence, there will be a net turning couple in the x^{μ}-frame:

$$\begin{aligned} \mathbf{T} &= \mathbf{x}_{\mathrm{a}} \times \mathbf{F}_{\mathrm{a}} + \mathbf{x}_{\mathrm{c}} \times \mathbf{F}_{\mathrm{c}} \\ &= (0, 0,\ F'l') + (0, 0, -F'l'(1-\beta^2)) = (0, 0,\ F'l'\beta^2)\ . \end{aligned} \tag{2.140}$$

In fact, Lewis and Tolman argued on the basis of the bent lever that the relativistic transformation law for forces should be such that it satisfies

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[82] One surmises that Laue's 'forces'-account of the Trouton-Noble experiment in Laue 1912b was at least partly inspired by his analysis of the thought experiment of Lewis and Tolman, even though the thought experiment is not mentioned in that paper.

[83] Tolman 1987, p. 79. Not surprisingly, given the important role his 1909 paper with Lewis played in Laue's work in 1911-1912, Tolman's textbook contains one of the best and most complete discussions of Laue's results available in the literature.

[84] See, e.g., Norton 1992a, p. 44.

$$\frac{F_x}{F_y} = \frac{1}{\gamma}\frac{F'_x}{F'_y}. \tag{2.141}$$

Only in that case would there be no net turning couple on the moving bent lever, something that seemed to be required by the relativity principle. However, as Sommerfeld pointed out to Laue, Eq. 2.141 is at odds with the relation

$$\frac{F_x}{F_y} = \gamma\frac{F'_x}{F'_y}, \tag{2.142}$$

that follows from the standard transformation law for forces, which is due to Planck (see Eq. 1.9) and of which Lewis and Tolman had apparently been unaware.

After our lengthy discussion of the Trouton-Noble experiment, one will immediately see how Laue could solve the apparent contradiction between the standard transformation law for forces and the turning couple it produces on the moving bent lever. The bent lever is stressed by the forces acting on it and therefore needs a turning couple to sustain uniform motion. This is just an instance of the Laue effect! In the less paradoxical terms that we have been using: the turning couple in Eq. 2.140 will be exactly compensated by another turning couple coming from the stresses in the bent lever.

Perhaps the easiest way to see where this compensating turning couple comes from is to invoke Planck's result that there is momentum associated to every energy flow (Laue 1911c, pp. 514–515). The force applied to the system at *a* and the reaction force to that force at *b* do work as the system is moving. As a consequence, the system steadily gains energy at *a* and steadily loses the same amount of energy at *b*. Hence, the argument continues, there is a constant energy flow from *a* to *b*. The momentum corresponding to this energy flow gives rise to a turning couple that is exactly opposite to the turning couple in Eq. 2.140.[85]

Both the energy flow and the momentum associated with it correspond to shear stresses in the arm *ab* in the bent lever's rest frame. To bring out the parallel with the Trouton-Noble condenser, I will derive the expression for the momentum density in the arm *ab* by transforming the energy-momentum tensor for the bent lever (not including the external forces) from the x'^{μ}-frame to the x^{μ}-frame.

The only non-vanishing components of the energy-momentum tensor in the x'^{μ}-frame are the shear stresses T'^{12} and T'^{21}:

$$T'^{12} = T'^{21} = F'/A', \tag{2.143}$$

[85] See Norton 1992a, p. 47, for a detailed version of this argument in modern notation.

where A' is the area of a cross section of the arm $a'b'$ (as indicated in Fig. 2.13). The energy-momentum tensor $T^{\mu\nu}$ in the frame in which the system is moving can be found through the Lorentz transformation

$$T^{\mu\nu} = \Lambda^{\mu}{}_{\rho}\, \Lambda^{\nu}{}_{\sigma}\, T'^{\rho\sigma} = \Lambda^{\mu}{}_{1}\, \Lambda^{\nu}{}_{2}\, T'^{12} + \Lambda^{\mu}{}_{2}\, \Lambda^{\nu}{}_{1}\, T'^{21}, \tag{2.144}$$

where $\Lambda^{\mu}{}_{\nu}$ is the matrix for a boost in the negative x-direction. For the (10, 20, 30)-components of $T^{\mu\nu}$ we find:

$$T^{10} = T^{30} = 0, \qquad T^{20} = \Lambda^{2}{}_{2}\, \Lambda^{0}{}_{1}\, T'^{21} = \gamma\, \beta\, (F'/A'). \tag{2.145}$$

Now use that these components divided by c and multiplied by the volume V of the arm ab of the bent lever are the components of the momentum in the arm ab. Call this momentum **P**:

$$\mathbf{P} = \tfrac{1}{c}\left(T^{10}, T^{20}, T^{30}\right)\mathsf{V} = \left(0,\ \frac{\beta\,\gamma\, F'\,\mathsf{V}}{c\,A'}, 0\right) = \left(0,\ \tfrac{1}{c}\,\beta\, F'\, l', 0\right), \tag{2.146}$$

where in the last step I used that $\gamma\ \mathsf{V}/A' = \mathsf{V}'/A' = l'$. The momentum **P** will give rise to a turning couple $-\,\mathbf{v} \times \mathbf{P}$ (cf. Eq. 1.38). When this turning couple is added to the turning couple in Eq. 2.140, we see that the net turning couple on the moving system vanishes just as the turning couple in the system's rest frame vanishes (see Eq. 2.139):

$$\begin{aligned} \mathbf{T} &= \mathbf{x}_a \times \mathbf{F}_a + \mathbf{x}_c \times \mathbf{F}_c - \mathbf{v} \times \mathbf{P} \\ &= (0, 0,\ F' l' \beta^2) - (0, 0,\ F' l' \beta^2) = (0, 0, 0)\,. \end{aligned} \tag{2.147}$$

The analogy between the Lewis-Tolman bent lever and the Trouton-Noble condenser will be obvious at this point. The external forces on the bent lever correspond to the Coulomb forces on the plates of the condenser and the shear stresses in the bent lever correspond to the non-electromagnetic stresses in the condenser. The Trouton-Noble experiment can be seen as a physical instantiation of the bent lever. Whittaker, for instance, writes immediately after his discussion of the bent lever: "This may be regarded as a model of the Trouton-Noble experiment" (Whittaker 1953, II, p. 56).

I want to make one final remark. The above analysis is in terms of the Laue picture. In the Rohrlich picture there will be no turning couples acting on the bent lever at all. I do not have the tools at this point to give a detailed analysis of the bent lever in the Rohrlich picture. I only developed the Rohrlich analysis of the Trouton-Noble experiment in terms of momentum and angular momentum, not in terms of forces. The easiest way to analyze the Lewis-Tolman bent lever in the Rohrlich picture, it seems to me, is to find the expression for the energy-momentum tensor $T'^{\mu\nu}$ giving the components F'^{i} of the external forces on the bent lever in its rest frame via the equation (cf. Eq. 2.117):

$$F'^{i} = -\int \partial'_j T'^{ij} \, d^3x'. \tag{2.148}$$

Using this energy-momentum tensor and the energy-momentum tensor describing the shear stresses in the bent lever (see Eqs. 2.143–2.145), one can then calculate the various contributions to the momentum and angular momentum of the system in the x^{μ}-frame under the Rohrlich definition and explicitly verify that there are no turning couples on the system. For my purposes, this calculation is not important and I will not go through it.

2.5.2 The mass-energy equivalence and the Trouton experiment. To conclude this chapter, I return to the original Trouton experiment, aimed at detecting an impulse upon charging or discharging a condenser. I will show that the Trouton experiment can be seen as a physical realization of a well-known thought experiment by Einstein, in the same way that the Trouton-Noble experiment can be seen as a physical realization of the thought experiment of Lewis and Tolman. However, whereas the relation between the Trouton-Noble experiment and the Lewis-Tolman bent lever was recognized early on by Laue, no one seems to have made the connection between the Trouton experiment and Einstein's thought experiment.[86] Drawing on the distinction between the Laue and the Rohrlich picture of what happens in a moving condenser, I will show what obscured this connection for the person most likely to see it, Lorentz.

About a year after he first introduced the inertia of energy (Einstein 1905b), Einstein showed that this relation between energy and mass is necessary and sufficient to ensure conservation of the motion of the center of mass for systems, in which, as Einstein put it, "not only mechanical, but also electromagnetic processes take place" (Einstein 1906, p. 627).[87] As Einstein acknowledges, his paper is similar to Poincaré 1900b, the latter's contribution to the Lorentz *Festschrift* of 1900 (see section 1.4). In his 1906 paper, Einstein argued that in order to avoid violations of the center of mass theorem of the kind Poincaré discussed in 1900, one has to assume that energy has inertia (Miller 1981, pp. 353–354; Darrigol 1994b, p. 3, pp.

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[86] I am indebted to John Stachel for drawing *my* attention to this connection.

[87] By 'mechanical processes' Einstein means non-electromagnetic processes that in the limit of low velocities can be described by ordinary Newtonian mechanics. Einstein's use of the term 'mechanics' is in accordance with the common usage of this term at the time (cf. Planck 1908, where 'mechanics' in this narrow sense is contrasted with 'general mechanics').

75–82[88]). Like Poincaré before him, he offered a vivid thought experiment in support of his claim.

In his lectures on relativity at the university of Leiden in 1910–1912, Lorentz used Einstein's thought experiment in his exposition of the inertia of energy (Lorentz 1922, pp. 242–243). Lorentz did not refer to Poincaré's earlier paper, nor to his discussions with Poincaré over Newton's third law in the context of which that paper was written. He did not mention the Trouton experiment either. The importance of these observations will become clear below.

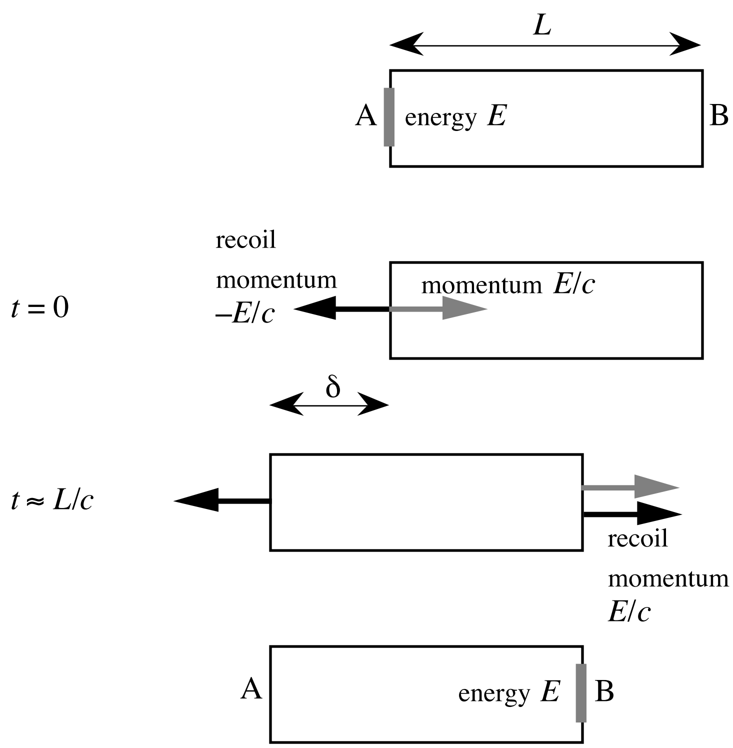

Figure 2.14 A thought experiment by Einstein to establish the equivalence of mass and energy.

Fig. 2.14 illustrates Einstein's thought experiment. Consider a box of mass M and length L in which the following process takes place. A small amount of energy E, stuck against the inside wall of the box at point A, is emitted in the form of electromagnetic radiation traveling to point B at the other end of the box. The radiation is absorbed, the energy is converted back to its original form, and ends up stuck against the inside wall of the box at B. On the assumption that E is massless, this experiment violates the center of mass theorem. The argument runs as follows. According to standard electrodynamics, the radiation has momentum E/c. Invoking

[88] Among other things, Darrigol reports that Langevin (1913, p. 414) claimed that he had independently found the derivation of $E = mc^2$ in Einstein 1906, and that he had lectured on this result in 1906 in Paris.

momentum conservation, the box will therefore recoil to the left with momentum E/c upon the emission of the radiation. When the radiation is re-absorbed, the box will recoil again, bringing it back to rest. While the radiation is in flight, for a period of approximately L/c, the box moves to the left with velocity E/cM. So, the box will move a distance

$$\delta \approx \frac{E}{c^2 M} L \tag{2.149}$$

during the course of the experiment, in violation of the center of mass theorem.

Einstein showed that this violation disappears if we assume that the energy E has mass $m = E/c^2$. In that case, the displacement to the left of the mass M of the box over the small distance $\delta \approx (m/M)L$ ($m \ll M$) is compensated by the displacement of the mass m of the energy E from the left to the right side of the box, and there is no net displacement of the center of mass of the system. Let δ' be the displacement of the center of mass due to the displacement of the mass m. With the help of Fig. 2.15, it is easy to compute δ' to the same approximation as Eq. 2.149 for δ.

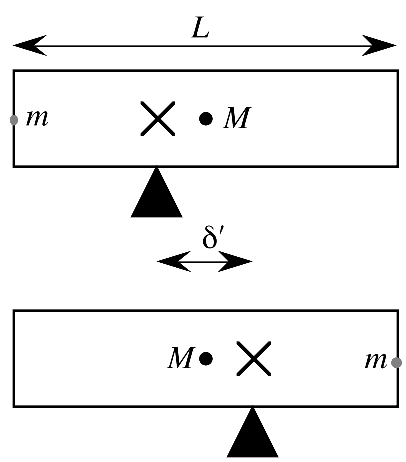

Figure 2.15 Balancing the box in Einstein's thought experiment on a wedge.

Imagine we carefully balance the system on a wedge supporting the box right under the center of mass of the system. We calculate the distance δ' that we have to move the wedge if the mass m moves from A to B. The condition δ' has to satisfy is:

$$M\frac{\delta'}{2} = m\frac{L-\delta'}{2} \approx m\frac{L}{2}. \tag{2.150}$$

It follows that $\delta' \approx (m/M)L$ which is indeed equal to δ in the same approximation. So, there is no net displacement of the center of mass. The equivalence of mass and energy prevents the center of mass theorem from being violated.[89]

Now consider the Trouton experiment again (cf. section 1.1, Figs. 1.1 and 1.2). Fig. 2.16 schematically shows the experimental setup, this time including the battery used for charging the condenser.

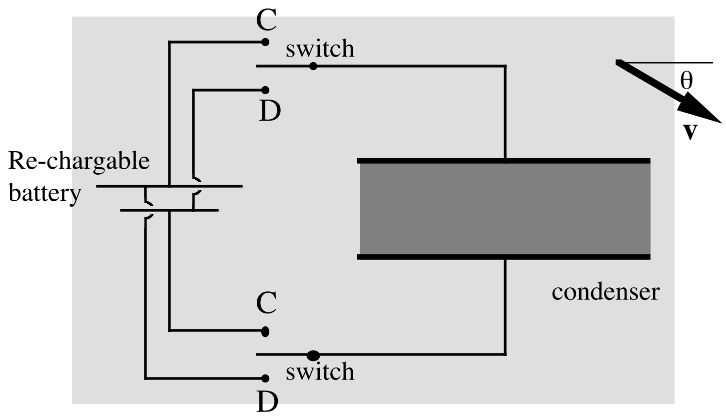

Figure 2.16 The Trouton experiment revisited.

The Trouton experiment is essentially equivalent to Einstein's thought experiment. Instead of having energy emitted as electromagnetic radiation which is then re-absorbed, we convert chemical energy of the battery to electromagnetic energy when we charge the condenser (i.e., when the switches in Fig. 2.16 are set to 'C'), which is then re-converted to chemical energy when we discharge the condenser and re-charge the battery (i.e., when the switches in Fig. 2.16 are set to 'D').[90] There are two complications.

First, since the electromagnetic field of a condenser will only carry momentum when the condenser is in motion, we need to consider the whole system in Fig. 2.16, condenser plus battery, in a frame in which it is moving. The importance of considering the case in which the

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[89] In his famous 1904 lecture in St. Louis, Poincaré came tantalizingly close to proclaiming the inertia of energy. Discussing a thought experiment essentially equivalent to the one discussed by Einstein in 1906, he writes: "Imagine, for example, a Hertzian oscillator, like those used in wireless telegraphy; it sends out energy in every direction; but we can provide it with a parabolic mirror, as Hertz did with his smallest oscillators, so as to send all the energy produced in a single direction. What happens then according to the theory? The apparatus recoils, as if it were a cannon and the projected energy a ball; and that is contrary to the [reaction] principle of Newton, *since our projectile here has no mass, it is not matter, it is energy*" (Poincaré 1904, p. 101; my italics). Poincaré discussed this same example at greater length, but without the tantalizing final remark, in his contribution to the Lorentz *Festschrift* (Poincaré 1900b, pp. 484–488).

[90] This last step obviously is not very realistic and is intended only to bring out the parallel with Einstein's thought experiment, in which we do not have to worry about the precise nature of the energy conversions at all.

condenser moves with its plates tilted at an angle θ with respect to its velocity will become clear below.

Second, in the Laue picture of the experiment in a frame in which the system is moving, the conversion of the battery's chemical energy to the energy of the condenser's electromagnetic field is not the only energy conversion we need to take into account. In the Laue picture, there is also energy associated with the electromagnetic and non-electromagnetic stresses in the condenser. Since the Laue picture and the Rohrlich picture are just alternative descriptions of the same process, this energy will have to come from the battery. In the Rohrlich picture, there is no energy associated with the stresses. Using the idealization of a perfectly rigid condenser, the conversion of the battery's chemical energy to the condenser's electromagnetic energy is the only conversion taking place in the Rohrlich picture.

Before I develop a detailed account of the experiment in both the Laue and the Rohrlich picture, I want to informally go through the analogue of Einstein's 1906 argument for the case of the Trouton experiment.

As in Einstein's thought experiment, the center of mass theorem will be violated in the Trouton experiment if the energy of the battery that is converted to energy of the condenser does not have mass. The argument runs as follows. When the condenser is charged, it gains momentum. Invoking momentum conservation, the system will recoil. When the condenser is discharged, it loses momentum. The system will recoil again, bringing it back to rest. In a frame in which the condenser is at rest before and after this experiment, the system will have moved to the left in the process, even though it is fully isolated. This is a violation of the center of mass theorem.

Of course, it suffices to look at the first stage of the experiment to see that we have a violation of the center of mass theorem. The violation occurs when the completely isolated system recoils upon having the battery charge the condenser. The equivalence of mass and energy prevents this violation of the theorem. To see how this comes about, we need to address the complication mentioned above that we have two different pictures of what happens in the experiment. To give a treatment that is valid in both we need to consider the condenser's total energy and its total momentum. These quantities are the same in the Laue and the Rohrlich picture. From Eq. 2.99 we read off that the total energy and total momentum of the condenser in the frame in which it is moving with velocity $\mathbf{v}$ is given by

$$U_{\text{tot}} = \gamma\, U'_{\text{tot}}, \qquad \mathbf{P}_{\text{tot}} = \gamma \left(U'_{\text{tot}}/c^2\right) \mathbf{v}\,, \tag{2.151}$$

where $U'_{\text{tot}} = U'_{\text{EM}} + U'_{\text{non-EM}}$. The non-electromagnetic energy $U'_{\text{non-EM}}$ in the condenser's rest frame will be the same whether the condenser is charged or not. It will be convenient to set it to zero. In that case, Eq. 2.151 directly gives the change in the energy and momentum of the condenser upon charging or discharging it.

This gives us all we need to show that the equivalence of mass and energy prevents the violation of the center of mass theorem in the Trouton experiment. As in Einstein's thought experiment, I will neglect terms of order β^2, although I want to emphasize that it is much more straightforward to make the argument exact in the case of the Trouton experiment than it is in the case of Einstein's thought experiment.

As the condenser is charged, an amount of energy $E = U_{\text{tot}} \approx U'_{\text{tot}}$ is transferred from the battery to the condenser. The condenser gains momentum $(E/c^2)\mathbf{v}$. If the energy E has mass $m = E/c^2$, the battery loses momentum $m\mathbf{v} = (E/c^2)\mathbf{v}$. So, the momentum of the system as a whole, condenser plus battery, is conserved. There will be no recoil. This is all that is needed in special relativity to give a detailed explanation of the negative result of the Trouton experiment. The experiment thus directly illustrates the theory's most famous equation, $E = mc^2$.

Why did Lorentz not see the connection between the Trouton experiment and Einstein's thought experiment of 1906? Given his own analysis of the Trouton experiment (Lorentz 1904b, pp. 194–196; see section 1.4) and given his discussions with Poincaré about Newton's third law, one would expect him to have seen this connection the moment he read Einstein's paper, which explicitly refers to Poincaré 1900b. Questions like this are notoriously hard to answer. It is often best to simply not ask them at all. It is always easy to see some connection with decades of hindsight. Still, this particular connection is so simple and Lorentz seemed so well primed to make it that the question why he missed it becomes legitimate. I think I have a plausible answer.

Recall that Lorentz only considered the electromagnetic momentum of the condenser, not its total momentum as we did above. In the Rohrlich picture, this does not make any difference, but in the Laue picture, which Lorentz, in effect, used, it does. In the Rohrlich picture, U_{tot} and $\mathbf{P}_{\text{tot}}$ in Eq. 2.151 are just U^{R}_{EM} and $\mathbf{P}^{\text{R}}_{\text{EM}}$; $U^{\text{R}}_{\text{non-EM}}$ and $\mathbf{P}^{\text{R}}_{\text{non-EM}}$ are zero once we set $U'_{\text{non-EM}}$ to zero (cf. Eqs. 2.114–2.115). In the Laue picture, on the other hand, U_{tot} and $\mathbf{P}_{\text{tot}}$ consist of an electromagnetic and a non-electromagnetic part, even when $U'_{\text{non-EM}} = 0$ (see Eqs. 2.104–2.105). To first order in β (i.e., $\gamma = 1$ and $\theta' = \theta$), these equations give (cf. Eq. 2.132 for the energy, and Eqs. 2.107–2.108 for the momentum):

$$U^{\text{L}}_{\text{EM}} = U'_{\text{EM}}, \qquad U^{\text{L}}_{\text{non-EM}} = 0,$$

$$\mathbf{P}^{\text{L}}_{\text{EM}} = 2\left(U'_{\text{EM}}/c^2\right) v \cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \tag{2.152}$$

$$\mathbf{P}^{L}_{\text{non-EM}} = \left(U'_{\text{EM}}/c^2\right) v \left[-\cos\theta \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix} + \sin\theta \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix} \right].$$

Lorentz (1904b, pp. 829–831) only considered the condenser's electromagnetic momentum. The argument I gave above in terms of the condenser's total energy and momentum does not work in terms of the condenser's electromagnetic energy and momentum alone. It does in the Rohrlich picture, but not in the Laue picture. If the energy U'_{EM} needed to charge the condenser (see Eq. 2.152) has mass U'_{EM}/c^2, the battery will lose momentum $(U'_{\text{EM}}/c^2)\mathbf{v}$. However, this does not compensate the condenser's gain in momentum which (see Eq. 2.152) is $2\cos\theta$ times that amount and is in the direction of the plates rather than in the direction of the velocity. So, it looks as if momentum conservation still requires the system to recoil, despite the assumption of the inertia of energy.

So, even Lorentz, the one person in our story with a keen understanding of both the Trouton experiment and Einstein's 1906 thought experiment, was in no position to make the connection between the two. Had Lorentz or someone else—say, Planck in 1908, or Laue in 1911—seen the connection, I think we can say, without much exaggeration, that the Trouton experiment would have become as famous as the Michelson-Morley experiment. Instead it has been all but forgotten.