MAX-PLANCK-INSTITUT FÜR WISSENSCHAFTSGESCHICHTE
Max Planck Institute for the History of Science

2012

PREPRINT 435

Jens Høyrup

Sanskrit-Prakrit interaction in elementary mathematics as reflected in Arabic and Italian formulations of the rule of three – and something more on the rule elsewhere

Sanskrit-Prakrit interaction in elementary mathematics as reflected in Arabic and Italian formulations of the rule of three – and something more on the rule elsewhere

Jens Høyrup
Roskilde University
Section for Philosophy and Science Studies
jensh@ruc.dk
http://ruc.dk/~jensh

Contribution to
International Seminar on History of Mathematics
Ramjas College, Delhi University
November 19-20, 2012

To be published in
Gaṇita Bhāratī

To the friends
Mahdi Abdeljaouad
and Ulrich Rebstock

Abstract

Sanskrit sources from Āryabhata to Bhāskara II have a standard formulation of the rule of three. However, it is clear that mathematics must also have been spoken of and performed during this period (and before) in vernacular environments, and that the two levels must have interacted – not least because the erudite astronomer-mathematicians use commercial arithmetic as the introduction to mathematics. But we have no surviving vernacular texts.

From Brahmagupta onward, however, the standard Sanskrit formulation is supplemented by the observation that two of the known magnitudes are similar in kind, and the third dissimilar. This could be an innovation made within the Sanskrit tradition, but comparison with Arabic and Italian medieval sources seems to rule this out. Instead, it must have been current in the commercial community spanning the Indian Ocean and the mediterranean – but since the Sanskrit scholars are not likely to have borrowed from Arabic traders, also in vernacular commercial arithmetic as practised within India.

So far, the story seems simple and coherent. However, if Latin twelfth-thirteenth-century writings and sources from the late medieval Ibero-Provencal area are taken into account, loose ends turn up that show the simple story not to be the whole story.

An introductory observation

Let me start with a necessary observation on terminology: *the "rule of three" is a rule, not a problem type*. It is a rule for solving linear problems of the type

if A corresponds to X, to what will B correspond?

The rule states in one way or the other (but with *this order* of the arithmetical operations) that the answer is $Y = (B \times X)/A$. Analysis of this "one way or the other" will be my main tool in what follows.

There has been a tendency among historians of mathematics to conflate *the rule* and the *problem type,* which has allowed them to find "the rule of three" in ancient Mesopotamia, ancient Egypt, and in the arithmetical epigrams of the *Greek Anthology*. The consequences of this can at best be understood through the folk tale motif of painting white crosses on all the doors of the town once the door of a suspect has been marked in that way: it ensures that the investigator will find nothing. So, I shall stick to etymology and reserve the name "rule" for the rule.

In consequence of this choice we do not find the rule of three in ancient Mesopotamian, Egyptian or Greek mathematics (or in any of those traditions that were directly derived from them before the Middle Ages). It belongs to ancient and medieval India and China, and (derived from India, as we shall see) to the medieval Arabic and Mediterranean world; from the Renaissance onward it also rises to fame in central and western Europe.

India

From Āryabhaṭa in the late fifth century CE onward, Sanskrit mathematicians use a standard terminology for the four magnitudes A, B, X, and Y – see, for instance, [Elfering 1975: 140] (Āryabhata), [Raṅgācārya 1912: 86] (Mahāvīra) and [Colebrooke 1817: 33, 283] (Bhāskara II, Brahmagupta):[1]

A: *pramāṇa* ("measure")	X: *phala* ("fruit")
B: *icchā* ("wish")	Y: *icchaphāla* ("fruit of wish")

[1] The Bakhshālī manuscript makes copious use of the rule (in particular for verifications) and refers to it by the usual name *trairāśika*; but the only time a partial terminology turns up (X 25, ed. [Hayashi 1995: 358, cf. 439]) it is clearly different.

There is, however, a much earlier Sanskrit appearance of the rule, albeit not making use of this terminology and therefore regarded by Sreeramula Rajeswara Sarma [2002: 135] as only "a rudimentary form of the Rule of Three". It is found in both recensions of the *Vedāṅgajyotiṣa* and may thus go back to c. 400 BCE [Pingree 1978: 536]. It states that the "known result is to be multiplied by the quantity for which the result is wanted, and divided by the quantity for which the known result is given", which as far as its arithmetic is concerned is not rudimentary at all. As pointed out by Sarma, the descriptive terms used – *jñāna⟨ta⟩rāśi*, "the quantity that is known", and *jñeya-rāśi*, "the quantity that is to be known" – also turn up in certain later texts.

All of these sources are written in Sanskrit – with the partial exception of the Bakhshālī manuscript, whose language "though intended to be Sanskrit, has been affected to a considerable degree by a dialect or dialects not only on the phonetic level but also on the morphologic level" [Hayashi 1995: 53], and which also (as mentioned) has a deviating terminology for the rule (and uses a standardized linear organization of the terms, which the Sanskrit sources may hint at but do not always draw). Āryabhaṭa as well as Brahmagupta present the rule within the context of astronomical treatises, and Bhāskara I and II were mainly astronomers. The very fact that the mathematics they introduce while having an astronomical purpose in mind is largely commercial or otherwise economical shows clearly, however, that it is borrowed from social groups that were distinct from that of the learned Brahmins and thus speakers of some Prakrit or other vernacular.[2] Mahāvīra, as a Jaina, was already part of an environment engaged in economical life [Thapar 1966: 65], and that exactly he would write a mathematical treatise not asked for by astronomy (though in the solemn language) fits the picture.[3] So does, finally, Bhāskara I's reference to "worldly practise (*lokavyavahāra*)" in connection with his discussion of the rule of three and elsewhere [Keller 2006: I, 107, cf. 12].[4] Here, as generally, to quote [Sarma 2010: 202], "Sanskrit has [...] absorbed much from the local traditions. Anthropologists recognize today that the so-called 'Little Traditions' played a significant role in shaping the 'Great Tradition'"

[2] Since distinctions and precision in this domain is already difficult for the specialist – cf. [Pollock 2006] – I, as a definite non-specialist, shall abstain from proposing any.

[3] That Mahāvīra was part of a distinct tradition is highlighted by the presence of several layers of Near Eastern / Mediterranean influence in his geometrical chapter – cf. [Høyrup 2004].

[4] Cf. also the reference to "worldly computations (*laukikagaṇita*)" when a problem about walking men is used to illustrate astronomical conjunction computation [Keller 2006: I, 127].

One thing is to deduct that vernacular mathematics must have existed. Another thing is to conclude anything about how it looked. In India as in other places where survival of (mathematical and other) texts relied on repeated copying, non-prestigious written culture had no better survival possibility than oral culture – that is, we depend almost exclusively on indirect evidence in the shape of references and quotations in the prestigious texts.[5]

Returning to the standard Sanskrit presentations of the rule of three, one feature may be a possible reference to vernacular ways (just *barely* possible when seen in the Indian context in isolation – but as we shall see, opening of the geographical horizon changes things). According to Brahmagupta [trans. Colebrooke 1817: 283],

> In the rule of three, argument, fruit and acquisition: the first and last terms must be similar.

Bhāskara I gives a somewhat related explanation in his commentary to the *Āryabhatīya:* not, however, when commenting upon Āryabhata's text but only in connection with the first example [ed., trans. Keller 2006: I, 109*f*], (the examples are Bhāskara's own contribution, the *Āryabhatīya* gives nothing but rules). With reference to the linear arrangement of the three known terms he states that

> the two similar (*sadṛśa*) (quantities) are at the beginning and the end. The dissimilar quantity (*asadṛśa*) is in the middle.

Mahāvīra [trans. Raṅgācārya 1912: 86] explains that

> in the rule-of-three, *Phala* multiplied by *Icchā* and divided by *Pramāṇa*, becomes the [required] answer, when the *Icchā* and the *Pramāṇa* are similar.

Bhāskara II [trans. Colebrooke 1817: 33, Sanskrit terms added], finally, states that

[5] One might hope that the strong reliance on memorization in Indian culture would improve the situation for the permanence of oral culture, but even memorization will probably have been reserved for prestigious cultural items – or at least have been selective, as illustrated by John Warren's observation in c. 1825 of a Tamil calendar maker who computed "a lunar eclipse by means of shells, placed on the ground, and from tables memorized 'by means of certain artificial words and syllables'" [Neugebauer 1952: 253]. It is next to certain that ethnomathematical field work will still be able to find surviving sub-scientific mathematical traditions (including their riddles), but the extent to which these are faithful in details to their first-millennium ancestors will be impossible to decide unless they can be connected to parallel sources, such as the "fragments of tables of multiplication, of squares and square roots, and of cubes and cube roots [which] are in Prakrit and must have been in use in the Andhra region at some time" in a Telugu commentary [Sarma 2010: 209] (tables agai, we observe); Sarma relates in parallel that "in Uttar Pradesh, elderly people tell me that they had memorized several multiplication tables of whole numbers and fractions in Vrajbhasa or in Avadhi" (still languages which belong to the second millennium – and perhaps in a form that belongs to the latest century).

> The first and last terms, which are the argument [*Pramāṇa*] and requisition [*Icchā*], must be of like denomination; the fruit [*Phala*], which is of a different species, stands between them.

Āryabhata had given no corresponding explanation in terms of the similar and the non-similar (nor does the Bakhshālī manuscript, but it is anyhow outside the main Sanskrit stream in terminology, as we have seen). It thus seems as if the concepts have been adopted into the tradition around the onset of the seventh century. The very different ways in which the Sanskrit authors insert the observation shows that they do not copy, one from the other.

That similarity is mentioned by Mahāvīra is a first argument that the concern with similarity originated in a vernacular environment (in economical transactions its relevance is obvious, in astronomical pure-number calculations less so); that Bhāskara I introduces the observation in connection with a (commercial) example points in the same direction. Neither argument is more than a non-compulsory hint, however; no wonder that those who have worked exclusively on Indian material have never been taken aback by the seventh-century introduction of what might be nothing but a reasonable mathematical observation.

Late medieval Italy

Things look different, however, if we take the Italian "abbacus" school and its mathematics into account. The abbacus school was a school mainly for merchants' and artisans' sons, who frequented it for two years or less around the age of 12, learning about calculation with Hindu-Arabic numerals and in general about basic commercial arithmetic – not least about the rule of three. The earliest references to the institution are from the 1260s, and the earliest textual witnesses of its mathematics from the outgoing thirteenth century.

One of the earliest formulations – perhaps the earliest one – presents the "rules of the three things"[6] as follows in literal translation [Arrighi 1989: 9, trans. JH]:

> If some computation was said to us in which three things are proposed, then we shall multiply the thing that we want to know with the one which is not of the same (kind),

[6] *Le regole delle tre cose* – plural because separate rules are given according to the absence or presence of fractions.

The treatise in question is a *Livero de l'abbecho*, known from a fourteenth-century copy in the manuscript Florence, Ricc. 2404 [ed. Arrighi 1989]. Because of misinterpretation of copied internal evidence, the treatise has been wrongly dated to 1288–90. It is likely to be somewhat but not very much later [Høyrup 2005: 27–28, 47].

and divide in the other.[7]

Exactly the same formulation of the rule (except that multiplication is "against", not "with") is found in an anonymous *Liber habaci* [ed. Arrighi 1987b: 111] which can be dated to c. 1310.[8] Already because this treatise uses no Hindu-Arabic but only Roman numerals (and fractions written with words), we may be sure that it is not derived from the *Livero*.

Two versions exist of Jacopo da Firenze's *Tractatus algorismi*, originally written in 1307 but known from three fifteenth-century copies. In one of these (Vatican, Vat. lat 4826) the rule of three is slightly more elaborate [ed., trans. Høyrup 2007: 236*f*, error corrected]:

> If some computation should be given to us in which three things were proposed then we should always multiply the thing that we want to know against the one which is not similar, and divide in the third thing, that is, in the other that remains.[9]

The first example given runs as follows (*tornesi* are minted in Tours, *parigini* in Paris):

> I want to give you the example to the said rule, and I want to say thus, vii *tornesi* are worth viiii *parigini*. Say me, how much will 20 *tornesi* be worth. Do thus, the thing that you want to know is that which 20 *tornesi* will be worth. And the not similar is that which vii *tornesi* are worth, that is, they are worth 9 *parigini*. And therefore we should multiply 9 *parigini* times 20, they make 180 *parigini*, and divide in 7, which is the third thing. Divide 180, from which results 25 and $^5/_7$. And 25 *parigini* and $^5/_7$ will 20 *tornesi* be worth.[10]

The other two manuscripts (Milan, Trivulziana MS 90, and Florence, Riccardiana MS 2236), probably representing a revised version [ed., trans. Høyrup 2007:], introduce the rule "If some computation should be *said* [*deta*] to us" (not "given"), while the rule itself turns around the final phrase, which becomes "... divide in

---

[7] Se ce fosse dicta alchuna ragione ella quale se proponesse tre chose, sì devemo moltiplicare quilla chosa che noie volemo sapere con quella che non è de quilla medessma, a partire nell'altra.

[8] Gino Arrighi's ascription to Paolo Gherardi (*fl.* 1328) is safely disregarded.

[9] Se ci fosse data alcuna ragione nela quale se proponesse tre cose, sì debiamo multiplicare sempre la cosa che noi vogliamo sapere contra a quella che non è simegliante,et parti nela terza cosa, cioè, nell'altra che remane.

[10] Vogliote dare l'exemplo ala dicta regola, et vo' dire chosì, vij tornisi vagliono viiij parigini. Dimmi quanto varranno 20 tornisi. Fa così, la cosa che tu voli sapere si è quello che varranno 20 tornisi. Et la non simegliante si è quello che vale vij tornisi, cioè, vagliono 9 parigini. Et però dobiamo multiplicare 9 parigini via 20, fanno 180 parigini, et parti in 7, che è la terza chosa. Parti 180, che ne viene 25 et $^5/_7$. Et 25 parigini et $^5/_7$ varrano 20 tornesi.

the other, that is, in the third thing";[11] their formulation of the example only differs from that of the Vatican manuscript by using the phrase "the one which is not of the same"[12] (that of the *Livero* etc.) instead of "the one which is not similar".

Jacopo, an emigrated Florentine, wrote his treatise in Montpellier. Paolo Gherardi wrote his *Libro di ragioni* in the same place in 1328. *His* formulation is that of the *Livero* and of the *Liber habaci*. So is that of Giovanni de' Danti's *Tractato d l'algorisimo* from 1370 [ed. Arrighi 1987a: 29], even though it copies much of its general introduction (a general praise of knowledge) from Jacopo.[13]

The examples of all these treatises differ; their shared formulation of the rule is thus not the consequence of one author copying from the other; it must represent a formulaic expression which was in general circulation. It remained so for long – it is still found in the first printed commercial arithmetic (*Larte de labbaco*, also known as the "Treviso arithmetic" from 1478[14]), while Luca Pacioli presents us with a slight pedagogical expansion in the *Summa de arithmetica* [1494: fol. 57^r, trans. JH], already present except for the words in {...} in his Perugia manuscript from 1478 [ed. Calzoni & Gavazzoni 1996: 19*f*]:

> The rule of 3 says that the thing which one wants to know is multiplied by that which is not similar, and divided by the other {which is similar}, and that which results will be of the nature of that which is not similar, and the divisor will always be of the similitude of the thing which one wants to know.[15]

In both cases an alternative follows:

> The rule of 3 says that the thing which is mentioned twice [*A* and *C* in the above letter formalism] should be looked for, of which the first is the divisor, and the second is multiplied by the thing mentioned once [*B*], and this multiplication is divided by the said divisor, and that which results from the said division will be of the nature of the thing mentioned once, and so much will the thing be worth {precisely} which we

[11] ... partire nel'altra, cioè nella terza cossa.

[12] quella che nonn'è di quella medesima.

[13] It is of course possible that Jacopo copies from an unknown earlier source, which might then just be shared by de' Danti (and the many others who have the same introduction). Given Jacopo's early date this is not very likely (but anyhow unimportant for our argument).

[14] Unpaginated, at least in my digital facsimile; but pp. 61*f* if the title page is page 1. In the end this treatise adds to the rule that the result will be of the nature of the non-similar thing, while the divisor will be similar.

[15] La regola del 3 vol che se multiplichi la cosa che l'homo vol saper per quella che non e simigliante e partire per l'altra {che e simigliante} e quel che ne vene si ene de la natura de quella che non è simigliante {e sira la valuta de la cosa che volemo inquirere}. E sempre el partitor convien che sia de la similitudine de la chosa che l'homo vol sapere.

try to know.[16]

Jacopo and Pacioli were not the only ones to insert pedagogical expansions. Another example is found in Pietro Paolo Muscharello's *Algorismus*, written in Nola[17] in 1478 [ed. Chiarini et al 1972: 59, trans. JH]:

> This is the rule of 3, which is the fundament for all commercial computations. And in order to find the divisor, always look for the similar thing, which is mentioned twice, and one of these will be the divisor, and I say that it will be the one which is not your request, and this your request you will get by multiplying with the other not similar thing, and this multiplication [i.e., product] you will have to divide by your divisor, and from it will come that which you will require.

As we see, Pacioli's reference to "the thing which is mentioned twice" is inserted here in the standard formula.

A last formulation to look at is found in Paolo dell'Abbaco's mid-fourteenth-century *Regoluzze* [ed. Arrighi 1966: 31, trans. JH], which does not formulate the rule as a merely arithmetical algorithm but prescribes a 2 × 2 organization on paper:

> If you want to calculate, that is, to make computations of sale and purchase, write the thing [*materia*] in front of its price, and the similar below the similar; and then multiply these two numbers that are askew, and always divide by the number which is beside.

As we see, not only the standard formula and its variations but also this practical prescription all circle around the concepts of the dissimilar and the similar.

Before we leave the Italian corpus, one weird aspect of the standard formula might be taken note of. The reference to *C* as "the thing that we want to know" is misleading: *C* itself is known, and that which we want to know is its counterpart (as made clear in the *Vedāṅgajyotisa*). But it would be the perfect translation of *icchā* or some corresponding vernacular Indian term; a linguistic loan is thus possible, though not very likely.

[16] Idem sub aliis verbis. La regola del 3 vol che se guardi la cosa mentovata doi volte de le quali la prima è partitore, e la seconda se moltiplica per la chosa mentoata una volta. E quella tal multiplicatione se parta per ditto partitore. E quello che ne vien de ditto partimento sira de la natura de la cosa mentovata una volta. E tanto varrà la chosa che cercamo sapere {aponto}.

[17] In Campania, and thus outside the native ground of the abbacus school, which may be the reason that it replaces the standard formula by an explanation.

Arabic sources

A first impression of the earliest extant Arabic description of the rule of three – the chapter on commercial transactions in al-Khwārizmī's *Algebra* from c. 820 CE – does not support any idea of transmission of formulations. It uses no name for the rule which might correspond to the Sanskrit or Italian reference to three things (actually, no name at all), and according to the best known translations it seems to build on the theory of proportions of *Elements* VII.

Frederic Rosen [1831: 68] translates as follows:

> You know that all mercantile transactions[18] of people, such as buying and selling, exchange and hire, comprehend always two notions and four numbers, which are stated by the enquirer; namely, measure and price, and quantity and sum. The number which expresses the measure is inversely proportionate to the number which expresses the sum, and the number of the price inversely proportionate to that of the quantity. Three of these four numbers are always known, one is unknown, and this is implied when the person inquiring says *how much?* and it is the object of the question. The computation in such instances is this, that you try the three given numbers; two of them must necessarily be inversely proportionate the one to the other. Then you multiply these two proportionate numbers by each other, and you divide the product by the third given number, the proportionate of which is unknown. The quotient of this division is the unknown number, which the inquirer asked for; and it is inversely proportionate to the divisor.

Roshdi Rashed agrees in his French translation [2007: 196] with Rosen that the translation must be made in terms of proportion theory but disagrees with Rosen in how to make the connection:

> Sache que toutes les transactions entre les gens, de vente, d'achat, de change <de monnaies>, de salaire, et toutes les autres, ont lieu selon deux modes, et d'après quatre nombres prononcés par le demandeur, qui sont: quantité d'évaluation, taux, prix, quantité évaluée.
>
> Le nombre qui est la quantité d'évaluation n'est pas proportionnel à celui qui est le prix. Le nombre qui est le taux n'est pas proportionnel au nombre de la quantité évaluée, et, parmi ces quatre nombres, trois sont toujours évidents et connus, et l'un d'eux est inconnu, qui, dans les termes de celui qui parle, est « combien », et qui est l'objet du demandeur.
>
> On l'infère ainsi ; tu examines les trois nombres évidents ; il est nécessaire que, parmi eux, il y en ait deux, dont chacun n'est pas proportionnel à son associé. Tu multiplies les deux nombres évidents non proportionnels l'un par l'autre; tu divises le produit par l'autre nombre évident, dont <l'associé> non proportionnel est inconnu ; ce que tu obtiens est le nombre inconnu cherché par le demandeur, et qui n'est pas proportionnel au nombre par lequel tu as divisé.

[18] The Arabic word is *muʿāmalāt*, referring to the economical transactions of social life in general, not only trade.

Where Rosen finds “inversely proportional”, Rashed thus sees “not proportional”. Neither makes much sense mathematically. The third modern translation, made by Boris Rozenfeld [1983: 45] therefore translates the critical term *mubāyin* neither as “not proportional” nor as “inversely proportional” but as *protiv*, “opposite”, probably thinking of a graphical 2x2-scheme as described by Paolo dell’Abbaco (and as found above, p. 1). Both twelfth-century Latin translations, due respectively to Robert of Chester [ed. Hughes 1989: 64] and Gerard of Cremona [ed. Hughes 1986: 255], do the same. In their time, indeed, this graphical scheme was well known, as made obvious by its use in the *Liber abbaci* (see below, n. 25).

Unfortunately, nothing in al-Khwārizmī’s text suggests that he knew about such a scheme – but fortunately, a much more meaningful translation of *mubāyin* can be given [Wehr 1985: 131] – namely “different (in kind)” (or “dissimilar”), as also indicated by Mohamed Souissi [1968: 96], with reference to precisely this passage. Al-Khwārizmī’s terminology is thus related both to what turns up in India from Bhāskara I and Brahmagupta onward and to what we find in late medieval Italy.

Quite a few later Arabic authors do refer to the Euclidean theory – sometimes integrating it with the presentation of the rule of three, sometimes keeping the two topics separate. Al-Karajī’s *Kāfī fi’l hisāb* (c. 1010 CE) is an example of separate treatment [ed., trans. Hochheim 1878: II, 15–17, English JH]:

> Chapter XLII. *Proportions*. Of the four magnitudes of the proportion, the first relates to the second as the third to the fourth. If you have found this correlation, then you obtain through interchange of the members that the first relates to the third as the second to the fourth. Further you also obtain, when combining, a proportion: the sum of the first and the second member relates to the second member as the sum of the third and the fourth member to the fourth. Further you may form differences [...].[19]
>
> If the first member is unknown, then you multiply the second by the third member and divide by the fourth. Similarly, if the fourth member is unknown, you divide this product by the first member. If the second or the third is unknown, then you multiply the first by the fourth and divide the product by the known one of the other two members.
>
> If three numbers form a proportion [...]
>
> Chapter XLIII. *Commercial transactions*. Know that in questions about commercial transactions you must have four magnitudes, which are pairwise similar, the price, the measure, the purchase amount and the quantity.[20]
>
> The price is the value of a measuring unit that is used in trade [...].
>
> [...] Of these four magnitudes, three are always known, and one is unknown. You find the unknown magnitude by multiplying one of the known magnitudes, for

[19] The set of operations performed here presupposes that all four magnitudes are of the same kind. Al-Karajī thus has the good reasons of a good mathematician to keep proportions and rule of three apart.

[20] Corresponding, respectively, to Rosen’s “price”, “measure”, “sum” and “quantity”.

> instance the sum or the quantity, by that which is dissimilar to it, namely the measure or the price, and dividing the outcome by the magnitude which is of the same kind. What comes from it is the result.
>
> Or if you prefer, put one of the known magnitudes, for instance the quantity or the sum [called the "purchase amount" a few lines earlier] in relation to the one that is similar to it [i.e., find their ratio], and thereby search the relation of the non-similar magnitude.[21]

It is clear already from the order of the four magnitudes involved (and corroborated by the whole formulation) that al-Karajī does not copy from al-Khwārizmī's exposition. This corresponds well to the presentation of algebra later in the treatise, which appears to draw on a pre-Khwārizmian form of that technique.[22]

Ibn al-Bannaʾ's concise *Talkhīs* (early 13th century CE) integrates the rule in the presentation of proportions (translated from [Souissi 1969: 87*f*]):

> The four proportional numbers are such that the first is to the second as the third to the fourth.
>
> The product of the first with the fourth is equal to the product of the second with the third.
>
> When multiplying the first by the fourth and dividing the product by the second, one obtains the third. [...]
>
> Whichever is unknown among these numbers can be obtained by this procedure from the other three, known, numbers. The method consists in multiplying the isolated given number, dissimilar from the two others, by that whose counterpart one ignores, and dividing by the third known number. The unknown results.

While al-Karajī and ibn al-Bannāʾ are usually counted as "mathematicians", ibn Thabāt was primarily a legal scholar, and his *Ghunyat al-Ḥussāb* ("Treasures of the Calculators", from around 1200 CE) is intended to teach the mathematics that could serve legal purposes. Even here, proportion theory and rule of three are integrated. The rule is stated thus (translated from [Rebstock 1993: 45]):

> The fundament for all *muʿāmalāt*-computation is that you multiply a given magnitude by one which is not of the same kind, and divide the outcome by the one which is of the same kind.

[21] In our letter symbols, $Y = ({}^{B}\!/_{A})\cdot X$. As we observe, this is not the rule of three but an alternative, here and in other Arabic works (e.g., ibn Thabāt, ed. [Rebstock 1993: 43]) called "by *niqba*", "by relation". It has the advantage of being intuitively easier to grasp, but the disadvantage of performing the division first, which will mostly increase rounding errors or entail difficult multiplications of fractional quantities.

[22] This follows in particular from his use of the key terms *al-jabr* and *al-muqābalah*, see [Høyrup 2007: 157], cf. [Saliba 1972].

This rule, as we see, coincides almost verbatim with the one that was taught in the Italian abbacus school. However, it precedes the earliest abbacus treatises by at least half a century, and in any case it is difficult to imagine that a scholar teaching in the Baghdad madrasa should have direct access to what went on in Italy. We may safely assume that the rule he knew was widespread in a commercial community spanning at least the whole region from Iraq to ibn al-Bannāʾ's Maghreb, and almost certainly also the traders of the Mediterranean as well as the Indian Ocean. Everywhere, it tended to penetrate even erudite presentations of the same subject-matter as a secondary explanation. *However, what penetrated the presentations of Brahmagupta, Mahāvīra and Bhāskara I and II can hardly have been the language of Arabic traders; it must have been the ways of autochthonous Indian merchants and public officials speaking a Prakrit.*

Latin presentations

This part of my story – the one that may convey information about Indian usages – turns out in the end to be, or at least to look, quite simple. However, this simplicity results from disregard of those features of the process that point away from it. Taking them into account will not refute the simple story, but they will show that there is more to the matter.

As we have seen, the two Latin translations of al-Khwārizmī's *Algebra* made in Iberian area in the twelfth century both misunderstand his reference to the similar and dissimilar, but apart from that they are faithful to the original. Two other twelfth-century Latin works from the same area, the *Liber mahamaleth* [ed. Vlasschaert 2010: II, 185, trans. JH] and the so-called "Toledan *Regule*" [ed. Burnett, Zhao & Lampe 2007: 155] have an approach which I know from nowhere else.[23] Of four numbers in proportion, the first and the fourth are declared "partners" (*socii*), and so are the second and the third. If one is unknown, then its partner shall divide any of the other two, and the outcome be multiplied by the third number – that is, the *nisba* approach or the seemingly similar rule $Y = ({}^{X}/_{A})\cdot B$.[24] Afterwards, both specify differently (without observing that there is a difference), namely in agreement with the naked rule of three,

[23] The two texts are generally related, see [Burnett, Zhao & Lampe 2007: 145]. Nothing forces us to believe that this shared peculiarity of theirs reflects a widespread pattern.

Even though the title *Liber mahamaleth* clearly shows that the work is intended to present the *muʿāmalāt* genre, this is not the only point where this treatise seems to explore matters on its own.

[24] This rule, corresponding to what was done in Mesopotamia, in Ancient Egypt and in Greek practical mathematics, will be in conflict with the Euclidean formulation in all practical applications, since A and X will not be "similar". It was therefore avoided by those Arabic authors who wanted to base their calculations on Euclid.

thus, if three are proposed and the fourth is unknown, multiply the second in the third, and divide what results by the first, and what comes out will be the fourth.

After a new headline "Chapter on buying and selling", the *Liber mahamaleth* [ed. Vlasschaert 2010: II, 186, trans. JH] repeats, but now recognizes that the methods are alternatives:

> When in buying or selling it is asked about something what is its price.
>
> Do thus: Multiply the middle [number] by the last, and divide the product by the first.
>
> Or divide the middle by the first, and what comes out of it multiply by the last, or divide the last by the first, and what comes out of it multiply by the middle. From all these modes results the unknown that is asked for.

The presentation of the matter in Fibonacci's *Liber abbaci* [ed. Boncompagni 1857: 83*f*, trans. JH] (from 1228, but at least this passage is likely to be close to the lost 1202 edition) looks like Fibonacci's personal way to describe what he has seen in use:

> In all commercial exchanges [*negotiationes – muʿāmalāt*?], four proportional numbers are always found, of which three are known, but the remaining unknown. The first of these three known numbers is the number of sale of any merchandise, be it number, or weight, or measure [explanatory examples]. The second, however, is the price of this sale [...]. The third, then, will be the sale of some quantity of this merchandise, whose price, namely the fourth, unknown number, will not be known. Therefore, in order to find the unknown number from those that are known, we give a universal rule for all cases, namely, in the top of a board write the first number to the right, namely the merchandise.[25] Behind in the same line you posit the price of the same merchandise, namely the second number. The third too, if it is the merchandise, write it under the merchandise, that is, under the first, And if it is the price, write it under the price, that is, under the second. In this way, as it is of the kind of that under which it is written, thus it will also be of the quality or the quantity, whether in number, in weight or in measure. This is, if the superior number, under which one is writing, is a number [of *rotuli*[26]], itself will also be *rotuli*, if pounds, pounds, [...]. When they are described thus, it will be obvious that two of those that are posited will always be contrary [*ex adverso*], which have to be multiplied together, and that if the outcome of their multiplication is divided by the third number, the fourth, unknown, will doubtlessly be found.

As we see, Fibonacci knows the reason for speaking of the similar and the dissimilar, but it does not enter his prescription (which it would strain normal

---

[25] This prescription corresponds to inscription on an Arabic dust- or clayboard (*takht* respectively *lawha*) – in agreement with the start from the right. Robert's and Gerard's translation of *mubāyin* as "opposite" shows that they think of the same scheme.

Comparing with Paolo's *Regoluzze* (above, p. 7), we observe that rows and columns are interchanged.

[26] A weight unit, see [Zupko 1981: 228].

language to call a “rule”).

Much later in the work, namely within the long chapter 12 consisting of mixed problems [ed. Boncompagni 1857: 170, trans. JH], we find a problem which is solved by means of the rule of three but which *prima facie* seems to have nothing to do with a general presentation of that rule:

> If it is asked about 6, to which number it has the same ratio [*proportio*] as 3 to 5, you do thus: Multiply 5 by 6, it will be 30; which divide by 3, 10 comes out of it, which is the number asked for; because as 3 is to 5, thus 6 is to 10. To be sure, we usually pose this question differently in our vernacular [*ex usu nostri vulgaris*]: namely that if 3 were 5, what then would 6 be? And just as it was said, 5 is similarly multiplied by 6, and the outcome divided by 3.

A similar problem follows, also given afterwards in “vernacular” terms. A total listing of the occurrences of the terms *vulgaris*/*vulgariter* in the work leaves no doubt that it refers to the usage of the precursor-environment for the abbacus school, the community of commercial calculators working around the Mediterranean.

Iberia and Provence

This may seem strange: so far we have encountered nothing similar to this presumed “vernacular” way. But this is only because we did not look at Ibero-Provençal material apart from what was written in Latin during the twelfth century, nor at what is probably the very earliest Italian abbacus book.

Disregarding chronology, let us start in 1482 with Francesc Santcliment’s Catalan *Suma de la art de arismetica*. It introduces the *regla de tres* in these words [ed. Malet 1998: 163, trans. JH]:

> It is called properly the rule of three, since within the said species 3 things are contained, of which two are similar and one is dissimilar. This said species is common to all sorts of merchandise. There is indeed no problem nor question, however tough it may be, which cannot be solved by it once it is well reduced.
>
> And in our vernacular [*nostre vulgar*] the said species begins: If so much is worth so much, what will so much be worth?
>
> The solution of this rule is commonly said: Multiply by its contrary and divide by its similar.

First, of course, we observe the reference to the “vernacular” connected to almost the same phrase (though no longer “counterfactual”, one thing “being worth” another one abstractly, not “being” a different thing). There are also references to the “similar” and the “dissimilar”, but the “common” formulation of the solution “Multiply by its contrary and divide by its similar” does not coincide precisely with the Italian standard abbacus rule. In spite of the shared reference to the vernacular, everything remains so different from the text of the *Liber abbaci* that any copying or direct inspiration from that work can be excluded.

A thorough inspection of all known commercial arithmetics of abbacus type written in Ibero-Provençal area until 1500 will show that they share the counterfactual or abstract "being-worth" formulation of the rule (now in chronological order).

The earliest of these treatises is a Castilian *Libro de arismética que es dicho alguarismo*, known from an early-sixteenth-century copy of an original written in 1393. Some aspects call to mind the *Liber mahamaleth*, showing the *Libro ... dicho alguarismo* to be partially rooted in an Iberian tradition going back to the Arabic period – especially use of "ascending composite fractions" ($^{a}/_{n}$ and $^{b}/_{p}$ of $^{1}/_{n}$ and ...). Most aspects, however, and in particular the presentation of the rule of three [ed. Caunedo del Potro & Córdoba de la Llave 2000: 147, trans. JH] are wholly different. This presentation combines the counterfactual with the abstract "being worth", and has no hint of a graphical organization in a 2x2-scheme (instead, the same linear organization is used as in the Bakhshālī manuscript, but this is too close at hand to be taken as evidence of any link):

> This is the 6th species, which begins "if so much is worth so much, what will so much be worth".
>
> Know that according to what the art of algorism commands, to make any calculation which begins in this way, "if so much was so much, what would so much be?", the art of algorism commands that you multiply the second by the third and divide by the first, and that which comes out of the division, that is what you ask for. As if somebody said, "if 3 were 4, what would 5 be?", in order to do it, posit the figures of the letters[27] as I say here, the 3 first and the 4 second and the 5 third, 3, 4, 5, and now multiply the 4, which is the second letter, with the 5, which is the third, and say, 4 times 5 are 20, and divide this 20 by the 3, which stands first, and from the division comes $6^{2}/_{3}$, so that if they ask you, "if 3 were 4, what would 5 be?", you will say $6^{2}/_{3}$, and by this rule all calculations of the world are made which are asked in this way, whatever they be.

Next in time comes the "Pamiers Algorism" from c. 1430 [Sesiano 1984: 27]. Jacques Sesiano offers a partial edition only, for which reason I cannot quote the whole introduction – but he does show [1984: 45] that it follows the pattern "$4^{1}/_{2}$ is worth $7^{2}/_{3}$, what is $13^{3}/_{4}$ worth?".

The anonymous mid–fifteenth-century Franco-Provençal *Traicté de la praticque d'algorisme* also follows the same general pattern but is never so close to the others that direct copying can be suspected. Its presentation of the rule of three [ed. Lamassé 2007: 469, trans. JH] runs thus:

> This rule is called rule of three for the reason that in the problems that are made by this rule three numbers are always required, of which the first and the third should always be similar by counting one thing. And from these three numbers result another one, which is the problem and conclusion of that which one wants to know. And it

[27] On p. 134 the author explains "the letters of algorism" to be the Hindu-Arabic numerals.

is always similar to the second number of the three. By some this rule is called the golden rule and by others the rule of proportions. The problems and questions of this rule are formed in this way: “If so much is worth so much, how much will so much be worth?”. As for example, “if 6 are worth 18, what would 9 be worth?”. For the making of such problems there is such a rule:

> Multiply that which you want to know by its contrary and then divide by its similar. Or multiply the third number by the second and then divide by the first.

As we see, this version emphasizes the similar and the dissimilar, and combines the linear arrangement of the Castilian *Libro de arismética* with the formulation we know from Santcliment.

Closely connected to this *Traicté* is Barthélemy de Romans' *Compendy de la praticque des nombres*.[28] It says about the rule of three [ed. Spiesser 2003: 255–257, trans. JH] that it is “the most profitable of all”, and gives two rules, one for finding *Y* from *A*, *X* and *B*, and one probably meant for finding *B* from *A*, *X* and *Y*,

> Multiply that which you want to know by its contrary, and then divide by its similar,

and

> Multiply that which you know by that which is wholly dissimilar to it, and then divide by its similar,

after which it goes on with the composites rules. The first of these rules, we see, is shared with the *Traicté* and with Santcliment; the second, by using the term dissimilar (*dissemblant*) instead of contrary, looks as if it was of Italian inspiration (it might thus simply bee an alternative formulation of the rule for finding *Y* from *A*, *X* and *B*). The first example, however, is in purely Iberian tradition, “if 5 is worth 7, what is 13 worth?”.

The final Ibero-Provençal treatise is Francés Pellos's *Compendion de l'abaco*, printed in Nice in 1492. It starts by a general introduction to the theme [ed. Lafont & Tournerie 1967: 101–103, trans. JH], that does not look in detail like anything else we have seen except in its last section, and which is likely to be Pellos's own description of the situation:

> This is the way how you should say in matters that ask: if so much is worth so much, how much is so much worth? In this way, you may understand more clearly in the following examples.
> *The first number.*
> The first number is always the thing bought or sold, and you need to keep it well in memory.
> *The second number.*
> Know that the second number shall always be the value or the price of that which you have bought or sold.

[28] Probably written around 1467 but only known from a revision made by Mathieu Préhoude in 1476.

The third example or number.

And the third number shall always be the thing that you want to know, that is to say, the thing that you want to by.

Remember that the first and the third numbers are always the same thing.

And know further that the first number and the third shall always be one thing. And if they are not certainly one thing, then you shall reduce them to a form where they speak of one thing, or matter, for in no way on earth they must not be different, as appears afterwards in the examples.

General rule to find every thing.

Always multiply the thing that you want to know by its contrary. And the outcome of this multiplication you divide by its similar, and that which comes out of such a division will be the value of the thing that you want to know.

The first examples that follow ask "if 4 are worth 9, what are 5 worth?", "if 3 and a half is worth 6, how much are 4 worth?", etc. After six similar examples follows a graphical scheme, deceptively similar to the one we know from Paolo dell'Abbacho and Fibonacci but actually used for reducing rule-of-three-type problems involving fractions into problems involving only integers (and too similar to many other schemes used in abbacus manuscripts to be supposed with any degree of certainty to be inspired by the traditional rule-of-three diagram).

Italy revisited

As we see, all Iberian and genuinely Procençal presentations of the rule use the counterfactual or abstract being-worth formulation; from the mid-fifteenth century *Traicté* onward they also know the notions of similar / dissimilar – whether because of interaction with the Italian tradition or because of other inspirations cannot be decided.

Many Italian abbacus treatises, on the other hand, also know the counterfactual problem – and even "counterfactual *calculations*", such as "If 9 is the 1/2 of 16, I ask you what part 12 will be of 25", found in Gherardi's *Libro di ragioni* [ed. Arrighi 1987b: 17, trans. JH]. But as was the case in the *Liber abbaci* (which by the way also contains counterfactual calculations), counterfactual problems and calculations are always found long after the presentation of the rule of three, or as illustrations of the rule following after many examples of the ordinary commercial kind.[29] Clearly, they are meant to be recreational, and do not belong to the basic didactical stock (for which reason they are invariably counterfactual, not of the abstract being-worth type).

There is *one* exception to this rule, namely the very earliest Italian abbacus treatise, the "Columbia Algorism" [ed. Vogel 1977], the original of which is likely

[29] This is where Jacopo [ed. trans. Høyrup 2007: 238] asks the question "if 5 times 5 would make 26, say me how much would 7 times 7 make at this same rate".

to have been written around 1285–90 (what we have is a fourteenth-century copy).[30] The rule of three is approached in two different ways.

On one hand, there is a general presentation of the rule, not mentioning the name "rule of three" [ed. Vogel 1977: 39*f*, trans. JH]:

> Remember, that you cannot state any computation where you do not mention three things; and it is fitting that one of these things must be mentioned by name two times; remember also that the first of the things that is mentioned two times by name must be the divisor, and the other two things must be multiplied together.

This is followed by an example dealing with the exchange of money. Later this formulation is used a couple of times [ed. Vogel 1977: 48, 50] in examples which explicitly speaks of the "rule of the three things". We recognize Pacioli's second formulation of the rule, which must thus have survived somewhere in the intervening two hundred years, even though I have not noticed it in texts I have looked at.

Mostly when the rule is used, however, a problem is reduced to a counterfactual [ed. Vogel 1977: 31*f*, 57, 61, 64*f*, 70 83, 80, 83, 86, 90, 109, 111, 123*f*] or an abstract being-worth [ed. Vogel 1977: 52, 112] formulation ("It is as if you said, 'if *a* were / is worth *b*, what ...'"). At times [ed. Vogel 1977: 52, 57, 64, 83], the rule is also mentioned by name in these connections.

Finally, a number of times the rule is called by name but without any reference to either the "mentioned", the counterfactual or the abstract being-worth formulations [ed. Vogel 1977: 52, 54*f*, 58, 110].[31] All in all, it is fairly obvious that the compiler of the treatise knows not only the Iberian "vernacular" way but also the idea underlying what was to become the Italian standard formula, expressing it however differently (the two "similar" things becoming that which is mentioned "two times").

The treatise stands at the very beginning of the Italian abbacus tradition, and it is thus not strange that it draws on discordant sources. In other respects too it has links to the Iberian tradition as we know it from the *Libro de arismética que es dicho alguarismo* – cf. for instance [Høyrup 2007: 85]. At the same time it makes use the Maghreb notation for ascending continued fractions (the *Libro ... dicho alguarismo* knows the fraction type but not the Maghrreb notation).[32] However,

[30] The dating of the treatise is discussed in [Høyrup 2007: 31 n. 20].

[31] The first folios of the treatise are missing, and so is the folio preceding the general introduction to the rule in "mentioned"-formulation. It is therefore impossible to exclude that a general introduction to the rule in one of the Ibero-Provençal formulations was present in the original. Nor must this necessarily have been the case, however.

[32] This notation was also used by Fibonacci, but Fibonacci is not a likely source. Firstly, Fibonacci always writes these fractions from right to left, as do the Maghreb writers; the writing direction in the Columbia algorism alternates. Secondly, the Columbia algorism

there are few traces in later times of its idiosyncrasies: it was copied at some moment during the fourteenth century; in 1344, Dardi of Pisa shares its (mis)use of the notation for ascending continued fractions, writing $\frac{2\ 1}{c\ 2}$ for "2 *censi* and ½ of a *censo*" [Høyrup 2010b: 23]; as we have seen, finally, the "mentioned" formulation of the rule of three turns up in two manuscripts from 1478 (Pacioli and Muscharello); but that is all I have observed. In particular, no Italian treatise I know of continued its use of the counterfactual or the abstract being-worth structures as the basic representation of or model for the rule of three.

What is the origin of the Ibero-Provençal representation?

As we have seen, Fibonacci refers to the counterfactual structure as the "vernacular" representation of the rule of three. Around 1200 it must hence have been widespread at least in some part of the Mediterranean environment he knew. Where?

We have no solids hints. For linguistic reasons, the counterfactual structure can hardly be Arabic: since the copula is not expressed in Semitic languages, "*a* is *b*, what is *c*?" should correspond to the opaque "*a b*, *c* what?". The abstract being-worth formulation, on the other hand, is obviously possible in Arabic, and we do have a couple of Arabic texts which come near to it. In the probably most faithful version of al-Khwārizmī's algebra – Gherardo's translation [ed. Hughes 1986: 256] – the first example deals with a commercial problem, "10 *qafīz* are for six dragmas, what do you get for four dragmas?"; the second, however, is abstract "'ten are for eight, how much is the price for four?´, or perhaps it is said, 'four of them are for which price'". The words "perhaps it is said" suggest that al-Khwārizmī quotes a common way of speaking – but we cannot be sure that this refers precisely to the abstract aspect of the formulation.

However, Robert's translation [ed. Hughes 1989: 65] as well as the extant Arabic version [ed. Rashed 2007: 198] have even the first example in abstract formulation. Both represent the text as it developed in use, and the change could thus reflect common parlance. On the other hand, the step from concrete to abstract formulation is easily made, and the very scattered occurrences of similar wordings in Arabic texts (and their absence from the *Liber mahamalet* and the "Toledan *regule*") do not suggest that they represent a widespread vernacular.[33]

sometimes uses the notation when *q* is not a denominator but a metrological unit, a thing Fibonacci would never do (he knows that a denominator is a divisor, not a denomination). Thirdly, the Columbia algorism has nothing else in common with the *Liber abbaci* (not to speak of Fibonacci's more sophisticated works).

[33] Ibn al-Khiḍr al-Qurašī (Damascus, mid-eleventh-century) explains [ed., trans. Rebstock 2001: 64, English JH] the foundation for "sale and purchase" to be the seventh book of

The safest assumption is that the abstract being-worth shape and its counterfactual variant were widespread at least somewhere in the Iberian world toward 1200;[34] before that we do not know.

So, whereas the first part of my story had few loose ends, these abound in the second part. Nobody should be surprised: loose ends are always there, and we can only create a simple coherent story by disregarding them – which does not make the coherent story untrue, only incomplete.

(And then I have not even touched at the presence of
the rule of three in China, another intriguing loose end.)

References

Arrighi, Gino, 1966. "La matematica del Rinascimento in Firenze: L'ereditá di Leonardo Pisano e le «botteghe d'abaco»". *Cultura e scuola* **5**:18, 287–294.

Arrighi, Gino (ed.), 1987a. Giovanni de' Danti Aretino, *Tractato de l'algorisimo*. Dal Cod. Plut. 30. 26 (sec. XIV) della Biblioteca Medicea Laurenziana di Firenze. *Atti e Memorie della Accademia Petrarca di Lettere, Arti e Scienze*, nuova serie **47**, 3–91.

Arrighi, Gino (ed.), 1987b. Paolo Gherardi, *Opera mathematica: Libro di ragioni – Liber habaci*. Codici Magliabechiani Classe XI, nn. 87 e 88 (sec. XIV) della Biblioteca Nazionale di Firenze. Lucca: Pacini-Fazzi.

Arrighi, Gino (ed.), 1989. "Maestro Umbro (sec. XIII), *Livero de l'abbecho*. (Cod. 2404 della Biblioteca Riccardiana di Firenze)". *Bollettino della Deputazione di Storia Patria per l'Umbria* **86**, 5–140.

Boncompagni, Baldassare, 1851. "Della vita e delle opere di Leonardo Pisano matematico del secolo decimoterzo". *Atti dell'Accademia pontificia de' Nuovi Lincei* **5** (1851–52), 5–9, 208–246.

Euclid, after which he states (emphasis added) that "this corresponds to *your* formulation, 'So much, which is known, for so much, which is known; how much is the price for so much, which is also known?'". The reference to "your formulation" suggests that a general way of speaking about the matter is referred to. But was this general way already abstract, or has the abstraction been superimposed by al-Qurašī?

In [Høyrup 2010a: 13], I took an isolated quotation from A. S. Saidan's translation of al-Baghdādī as a hint of Persian counterfactual usage. Since then Mahdi Abdeljaouad has got access to the book and translated the whole surrounding passage (for which my sincere thanks!). In context, the use of the Persian expressions *dah yazidah*, "ten (is) eleven", and *dah diyazidah*, "ten (is) twelve" in calculations of profit and loss turn out to have no such implications. Cognitively, what goes on is rather an analogue of the medieval Italian (and modern) notion of *per cento*, "percent".

[34] It should be remembered that the 13th-century Vatican manuscript Palat. 1343 of the *Liber abbaci* refers (fol. $47^{r–v}$, new foliation) to use of a work by a "Castilian master". Boncompagni knew [1851: 32], but since this early manuscript is incomplete, he used a later one (from where this observation is absent) for his edition.

Boncompagni, Baldassare (ed.), 1857. *Scritti* di Leonardo Pisano matematico del secolo decimoterzo. I. Il *Liber abbaci* di Leonardo Pisano. Roma: Tipografia delle Scienze Matematiche e Fisiche.

Burnett, Charles, Ji-Wei Zhao & Kurt Lampe, 2007. "The Toledan *Regule:* a Twelfth-Century Arithmetical Miscellany". *SCIAMUS* **8**, 141–231.

Calzoni, Giuseppe, & Gianfranco Cavazzoni (eds), 1996. Luca Pacioli, *"tractatus mathematicus ad discipulos perusinos"*. Città di Castello: Delta Grafica.

Caunedo del Potro, Betsabé, & Ricardo Córdoba de la Llave (eds), 2000. *El arte del alguarismo*. Un libro castellano de aritmética comercial y de ensayo de moneda del siglo XIV. (Ms. 46 de la Real Colegiato de San Isidoro de León). Salamanca: Junta de Castilla y León, Consejeria de Educación y Cultura.

Chiarini, Giorgio, et al (eds), 1972. [Pierpaolo Muscharello], *Algorismus. Trattato di aritmetica pratica e mercantile del secolo XV*. 2 vols. Verona: Banca Commerciale Italiana.

Colebrooke, H. T. (ed., trans.), 1817. *Algebra, with Arithmetic and Mensuration from the Sanscrit of Brahmagupta and Bhascara*. London: John Murray.

Elfering, Kurt (ed., trans.), 1975. *Die Mathematik des Aryabhata I: Text, Übersetzung aus dem Sanskrit und Kommentar*. München: Wilhelm Fink.

Hayashi, Takao, 1995. *The Bakhshālī Manuscript: An Ancient Indian Mathematical Treatise*. Groningen: Egbert Forsten.

Hochheim, Adolf (ed., trans.), 1878. *Kâfî fîl Hisâb (Genügendes über Arithmetik)* des Abu Bekr Muhammed ben Alhusein Alkarkhi. I-III. Halle: Louis Nebert.

Høyrup, Jens, 2004. "Mahāvīra's Geometrical Problems: Traces of Unknown Links between Jaina and Mediterranean Mathematics in the Classical Ages", pp. 83–95 *in* Ivor Grattan-Guinness & B. S. Yadav (eds), *History of the Mathematical Sciences*. New Delhi: Hindustan Book Agency.

Høyrup, Jens, 2005. "Leonardo Fibonacci and *Abbaco* Culture: a Proposal to Invert the Roles". *Revue d'Histoire des Mathématiques* **11**, 23–56.

Høyrup, Jens, 2007. *Jacopo da Firenze's Tractatus Algorismi and Early Italian Abbacus Culture*. Basel etc.: Birkhäuser.

Høyrup, Jens, 2010a. "Fibonacci – Protagonist or Witness? Who Taught Catholic Christian Europe about Mediterranean Commercial Arithmetic?" Paper presented at the workshop "Borders and Gates or Open Spaces? Knowledge Cultures in the Mediterranean During the 14th and 15th Centuries", Departamento de Filosofia y Lógica, Universidad de Sevilla, 17–20 December 2010. *Preprint*, Roskilde University, 26 December 2010. http://www.akira.ruc.dk/~jensh/publications/2010%7Be%7D_Fibonacci--Protagonist%20or%20Witness_preprint.PDF

Høyrup, Jens, 2010b. "Hesitating progress – the slow development toward algebraic symbolization in abbacus- and related manuscripts, c. 1300 to c. 1550", pp. 3–56 *in* Albrecht Heeffer & Maarten Van Dyck (eds), *Philosophical Aspects of Symbolic Reasoning Early in Modern Mathematics*. London: College Publications.

Hughes, Barnabas, O.F.M., 1986. "Gerard of Cremona's Translation of al-Khwārizmī's *Al-Jabr*: A Critical Edition". *Mediaeval Studies* **48**, 211–263.

Hughes, Barnabas B. (ed.), 1989. Robert of Chester's Latin translation of al-Khwārizmī's *Al-jabr*. A New Critical Edition. Wiesbaden: Franz Steiner.

Keller, Agathe, 2006. *Expounding the Mathematical Seed.* A Translation of Bhāskara I on the Mathematical Chapter of the *Āryabhaṭīya*. 2 vols. Basel etc.: Birkhäuser.

Lafont, Robert, & Guy Tournerie (eds), 1967. Francès Pellos, *Compendion de l'abaco*. Montpellier: Édition de la Revue des Langues Romanes.

Lamassé, Stéphane, 2007. "Les problèmes dans les arithmétiques commerciales en langue française et occitane de la fin du Moyen Âge". *Thèse de doctorat*, Université de Paris 1 Panthéon-Sorbonne, novembre 2007. I thank Stéphane Lamassé for the permission to cite his unpublished dissertation.

Malet, Antoni (ed.), 1998. Francesc Santcliment, *Summa de l'art d'Aritmètica*. Vic: Eumo Editorial.

Neugebauer, Otto, 1952. "Tamil Astronomy: A Study of the History of Astronomy in India". *Osiris* **10**, 252–267.

Pacioli, Luca, 1494. *Summa de Arithmetica Geometria Proportioni et Proportionalita*. Venezia: Paganino de Paganini.

Pingree, David, 1978. "History of Mathematical Astronomy in India", pp. 533–633 *in Dictionary of Scientific Biography*, vol. XV. New York: Scribner.

Pollock, Sheldon, 2006. *The Language of the Gods in the World of Men: Sanskrit, Culture, and Power in Premodern India*. Berkeley etc.: University of California Press.

Raṅgācārya, M. (ed., trans.), 1912. The *Gaṇita-sāra-sangraha* of Mahāvīrācārya with English Translation and Notes. Madras: Government Press.

Rashed, Roshdi (ed., trans.), 2007. Al-Khwārizmī, *Le Commencement de l'algèbre*. Paris: Blanchard.

Rebstock, Ulrich, 1993. *Die Reichtümer der Rechner (Ġunyat al-Ḥussāb) von Aḥmad b. Ṯabāt (gest. 631/1234). Die Araber – Vorläufer der Rechenkunst*. Walldorf-Hessen: Verlag für Orientkunde Dr. H. Vorndran.

Rebstock, Ulrich (ed., trans.), 2001. ʿAlī ibn al-Ḫiḍr al-Qurašī (st. 459 / 1067), *At-taḏkira bi-uṣūl al-ḥisāb wa l'farā'iḍ* (*Buch über die Grundlagen der Arithmetik und der Erbteilung*). Frankfurt a.M.: Institute for the History of Arabic-Islamic Science at the Johann Wolfgang Goethe University.

Rosen, Frederic (ed., trans.), 1831. The *Algebra* of Muhammad ben Musa, Edited and Translated. London: The Oriental Translation Fund.

Rozenfeld, Boris A. (ed. trans.), 1983. Muhammad ibn Musa al-Xorezmi, *Kratkaja kniga ob isčiclenii algebry i almukabaly*, pp. 20–81, notes 118–142 *in* S. X. Siraždinov (ed.), Muxammad ibn Musa al-Xorezmi *Matematičeskie traktaty*. Taškent: Izdatel'stvo »FAN« Uzbekskoj CCP.

Saliba, George A., 1972. "The Meaning of al-jabr wa'l-muqābalah". *Centaurus* **17** (1972-73), 189–204.

Sarma, Sreeramula Rajeswara, 2002. "Rule of Three and Its Variations in India", pp. 133–156 *in* Yvonne Dold-Samplonius et al (eds), *From China to Paris: 2000 Years Transmission of Mathematical Ideas*. Stuttgart: Steiner.

Sarma, Sreeramula Rajeswara, 2010. "Mathematical Literature in the Regional Languages of India", pp. 201–211 *in* B. S. Yadav (ed.), *Ancient Indian Leaps in the Advent of mathematics*. Basel: Birkhäuser.

Sesiano, Jacques, 1984. “Une arithmétique médiévale en langue provençale”. *Centaurus* **27**, 26–75.

Souissi, Mohamed, 1968. *La langue des mathématiques en arabe*. Thèse principale pour le Grade de Docteur d’État, Université de Paris, Faculté des lettres et des sciences humaines. Tunis: Université de Tunis.

Souissi, Mohamed (ed., trans.), 1969. Ibn al-Bannā’, *Talkhīṣ aʿmāl al-ḥisāb*. Texte établi, annoté et traduit. Tunis: L’Université de Tunis.

Spiesser, Maryvonne (ed.), 2003. *Une arithmétique commerciale du XV^e^ siècle*. Le *Compendy de la praticque des nombres* de Barthélemy de Romans. Turnhout: Brepols.

Thapar, Romila, 1966. *A History of India*. Vol. I. Harmondsworth, Middlesex: Penguin.

Vlasschaert, Anne-Marie (ed.), 2010. *Liber Mahamaleth*. Wiesbaden: Franz Steiner.

Vogel, Kurt, 1977. *Ein italienisches Rechenbuch aus dem 14. Jahrhundert (Columbia X 511 A13)*. (Veröffentlichungen des Deutschen Museums für die Geschichte der Wissenschaften und der Technik. Reihe C, Quellentexte und Übersetzungen, Nr. 33). München: Deutsches Museum.

Wehr, Hans, 1985. *Arabisches Wörterbuch für die Schriftsprache der gegenwart. Arabisch-Deutsch*. 5. Auflage. Wiesbaden: Harrassowitz.

Zupko, Ronald Edward, 1981. *Italian Weights and Measures from the Middle Ages to the Nineteenth Century*. Philadelphia: American Philosophical Society.

MAX-PLANCK-INSTITUT FÜR WISSENSCHAFTSGESCHICHTE

Max Planck Institute for the History of Science

Preprints since 2010 (a full list can be found at our website)

391 Horst-Heino v. Borzeszkowski & Renate Wahsner **Die Fassung der Welt unter der Form des Objekts und der philosophische Begriff der Objektivität**

392 Ana Barahona, Edna Suarez-Díaz, and Hans-Jörg Rheinberger (eds.) **The Hereditary Hourglass. Genetics and Epigenetics, 1868–2000**

393 Luis Campos and Alexander von Schwerin (eds.) **Making Mutations: Objects, Practices, Contexts**

394 Volkmar Schüller **Some Remarks on Prop. VIII Probl. II of Newton's Opticks Book I Part I**

395 Tamás Demeter **Hume's Experimental Method**

396 Fynn Ole Engler, Björn Henning und Karsten Böger **Transformationen der wissenschaftlichen Philosophie und ihre integrative Kraft – Wolfgang Köhler, Otto Neurath und Moritz Schlick**

397 Frank W. Stahnisch **«Der Rosenthal'sche Versuch» oder: Über den Ort produktiver Forschung – Zur Exkursion des physiologischen Experimentallabors von Isidor Rosenthal (1836–1915) von der Stadt aufs Land**

398 Angela Matyssek **Überleben und Restaurierung. Barnett Newmans *Who's afraid of Red, Yellow, and Blue III* und *Cathedra***

399 Susanne Lehmann-Brauns, Christian Sichau, Helmuth Trischler (eds.) **The Exhibition as Product and Generator of Scholarship**

400 Fynn Ole Engler und Jürgen Renn **Wissenschaftliche Philosophie, moderne Wissenschaft und Historische Epistemologie**

401 M. J. Geller **Look to the Stars: Babylonian medicine, magic, astrology and *melothesia***

402 Matthias Schemmel **Medieval Representations of Change and Their Early Modern Application** (TOPOI – Towards a Historical Epistemology of Space)

403 Frank W. Stahnisch **German-Speaking Émigré Neuroscientists in North America after 1933: Critical Reflections on Emigration-Induced Scientific Change**

404 Francesca Bordogna **Asceticism and Truth: The Case of 'Magic Pragmatism'**

405 Christoph Hoffmann and Alexandre Métraux (eds.) **Working with Instruments – Three Papers of Ernst Mach and Ludwig Mach** (Translated by Daniel Bowles)

406 Karin Krauthausen **Paul Valéry and Geometry: Instrument, Writing Model, Practice**

407 Wolfgang Lefèvre **Picturing the World of Mining in the Renaissance The *Schwazer Bergbuch* (1556)**

408 Tobias Breidenmoser, Fynn Ole Engler, Günther Jirikowski, Michael Pohl and Dieter G. Weiss **Transformation of Scientific Knowledge in Biology: Changes in our Understanding of the Living Cell through Microscopic Imaging**

409 Werner Kogge **Schrift und das Rätsel des Lebendigen.** Die Entstehung des Begriffssystems der Molekularbiologie zwischen 1880 und 1950

410 Florentina Badalanova Geller ***2 (Slavonic Apocalypse of) Enoch*: Text and Context**

411 Henning Schmidgen & Urs Schoepflin (eds.) **Hans-Jörg Rheinberger : a Bibliography**

412 Renate Wahsner & Horst-Heino v. Borzeszkowski **Erkenntnis statt Erbauung: Hegel und das naturwissenschaftliche Denken der Moderne**

413 Mirjam Brusius **From photographic science to scientific photography: Photographic experiments at the British Museum around 1850**

414 Viktor J. Frenkel **Professor Friedrich Houtermans – Arbeit, Leben, Schicksal. Biographie eines Physikers des zwanzigsten Jahrhunderts.** Herausgegeben und ergänzt von Dieter Hoffmann, unter Mitwirkung von Mary Beer

415 Ilana Löwy (ed.) **Microscope Slides – Reassessing a Neglected Historical Ressource**

416 André L. Blum, John Michael Krois und Hans-Jörg Rheinberger (Hrsg.) **Verkörperungen**

417 Pietro Daniel Omodeo **Sixteenth Century Professors of Mathematics at the German University of Helmstedt.** A Case Study on Renaissance Scholarly Work and Networks

418 Peter Schöttler & Hans-Jörg Rheinberger (éds.) **Marc Bloch et les crises du savoir**

419 Albert Presas i Puig (ed.) **A Comparative Study of European Nuclear Energy Programs**

420 Mathias Grote & Max Stadler (eds.) **Membranes Surfaces Boundaries**
Interstices in the History of Science, Technology and Culture

421 Frank W. Stahnisch **The emergence of *Nervennahrung*: Nerves, mind and metabolism in the long eighteenth century**

422 Pietro Daniel Omodeo, Irina Tupikova **Aristotle and Ptolemy on Geocentrism: Diverging Argumentative Strategies and Epistemologies** (TOPOI – Towards a Historical Epistemology of Space)

423 Han F. Vermeulen **Linguistik und Völkerkunde – der Beitrag der historisch-vergleichenden Linguistik von G.W. Leibniz zur Entstehung der Völkerkunde im 18. Jahrhundert**
[Leicht erweiterte Fassung des Working Papers No. 133 aus dem MPI for Social Anthropology]

424 Alfred Gierer **Mit Schiller gegen den „Egoismus der Vernunft“.** Zeitübergreifende Gedanken zur Natur des Menschen

425 Annette Vogt **Die Berliner Humboldt–Universität von 1945/1946 bis 1960/1961**

426 Klaus Geus, Martin Thiering (eds.) **Common Sense Geography and Mental Modelling**

427 Renate Wahsner **Kann eine moderne Naturphilosophie auf Hegelsche Prinzipien gegründet werden?** Spekulatives und naturwissenschaftliches Denken

428 Stefano Bordoni **Widening the Scope of Analytical Mechanics** Duhem's third pathway to Thermodynamics

429 Pietro Daniel Omodeo **Copernicus in the Cultural Debates of the Renaissance: Reception, Legacy, Transformation** [Part I & II]

430 Mark Geller & Klaus Geus (eds.) **Productive Errors: Scientific Concepts in Antiquity**
(TOPOI – Dahlem Seminar for the History of Ancient Sciences)

431 Klaus Gottstein **The Amaldi Conferences. Their Past and Their Potential Future**

432 Mikuláš Teich **The Scientific Revolution Revisited**

433 Lorraine Daston & Jürgen Renn (Hrsg.) **Festkolloquium für Hans-Jörg Rheinberger**
Beiträge zum Symposium am 24. 1. 2011 im Max-Planck-Institut für Wissenschaftsgeschichte

434 Conference **Epistemology and History. From Bachelard and Canguilhem to Today's History of Science**