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Canon and commentary in ancient China: An outlook based on mathematical sources

CANON AND COMMENTARY IN ANCIENT CHINA: AN OUTLOOK BASED ON MATHEMATICAL SOURCES

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Some decades after the unification of the Chinese Empire, the compilation of a mathematical book, *The Nine Chapters on Mathematical Procedures* (*Jiuzhang suanshu*) (hereafter *The Nine Chapters*), that was to have a singular fate in China began. A few centuries after its completion in the first century BCE or CE,[1] the book came to be referred to as a "Canon" (*jing*), and even, later on during the Song dynasty, as the most important of all mathematical Canons.

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[*] This paper is the reworked version of a talk presented at a workshop organized by Professor Kim Yung Sik on the topic "Critical Problems in the History of East Asian Science" and held at the Dibner Institute November 16–18, 2001. It is my pleasure to thank Professor Kim Yung Sik for having invited me to take part in the workshop and for his remarks on a first version of this paper. I am also grateful for the comments Sir Professor Geoffrey Lloyd sent me, which helped clarify some points that had remained obscure. Last but not least, I wish to express my thanks to all the participants in the workshop and to Bruno Belhoste for the discussions that helped me improve my understanding of the problem tackled in this paper. The final version was completed during the stay I made at the Max Planck Institut für Wissenschaftsgeschichte, in the summer 2007. I benefited from last remarks by Alexei Volkov, whom I am delighted to thank here. Kelaine Vargas played a key part in making the argument clearer. I have pleasure in thanking her wholeheartedly. Since this paper simultaneously addresses various readerships, including sinologists and historians of science, both communities might find some of the arguments too lengthy for their own use. I apologize for any impatience caused to the reader, for the too many details given in cases that would seem to some to require none.

[1] Diverging views about the completion of *The Nine Chapters* are still under discussion today. I have argued elsewhere in favor of dating the completion of the compilation to the first century CE. (See my introduction to chapter 6, in Karine Chemla & Guo Shuchun, *Les Neuf chapitres. Le classique mathématique de la Chine ancienne et ses commentaires* [Paris: Dunod, 2004], pp. 475–478. In Chapter B of Chemla & Guo, *Les Neuf chapitres*, pp. 43–56, Guo Shuchun presents the evolution of scholars' views on this issue through history). For further detail about the status of the book as a Canon, see below. Together with Prof. Guo Shuchun (Institute for the History of Natural Sciences, Academy of Sciences, Beijing, China) between 1984 and 2004, I worked on the critical edition and the French translation of *The Nine Chapters* and the commentaries published in Chemla & Guo, *Les Neuf chapitres*. My ideas on these sources were deeply influenced by our collaboration.

However insignificant this may seem, it raises important issues. For centuries, Chinese readers thus perceived the book as belonging to a specific set of scriptures, all designated by the technical term *jing*. What did it mean for a book to be perceived as a Canon? How was it read accordingly? What kind of attitude, of expectation, did this fact induce in the readers? These are the questions that constitute the scope of this essay. They are much too broad to be exhaustively addressed within a few pages. I shall therefore restrict myself to tackling them from a specific angle, hoping to demonstrate thereby how fruitful they could be for different directions of research.

In fact, in his *Scripture, Canon and Commentary. A Comparison of Confucian and Western Exegesis*, John Henderson treated these questions extensively, mainly with reference to the Confucian Canons. As scripture, these texts have all been the object of traditions of commentary, and, in an attempt to capture how readers perceived them, Henderson observed the attitudes and expectations manifested by commentators towards the Confucian Canons. His survey offers an appropriate background for identifying which attitudes towards *The Nine Chapters* documented in our historical sources could stem from perceiving the book as a "Canon." Two issues appear to me to be at stake when reconsidering these questions with respect to a mathematical text.

The first one relates to the history of texts. Despite pioneering research about perceptions of Canons and modes of reading them as well as about traditions of commentary,[2] we are far from a comprehensive understanding of the properties attributed to this category of texts and the approaches to them developed by readers throughout Chinese

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[2] John B. Henderson, *Scripture, Canon and Commentary. A Comparison of Confucian and Western Exegesis* (Princeton: Princeton University Press, 1991) aimed at exploring the assumptions on Canons shared by commentators, not only in Chinese history, but in a comparative perspective. In contrast, Rudolf Wagner developed a project devoted to commentaries as writings *per se*, in their variety. In his *The Craft of a Chinese Commentator. Wang Bi on the* Laozi, (New York: State University of New York Press, 2000), Wagner particularly concentrates on Wang Bi's commentary on the *Daodejing*, and, incidentally, on the *Yijing*. Wagner thereby extends the type of Canon examined beyond the strict circle of the Confucian scripture. His focus lies in describing ways in which Wang Bi as a commentator interprets the Canons and how his philosophy takes shape as commentary. It is interesting to note that aspects of representations of how the Canon makes sense are common to this case and to ours (see for instance Rudolph Wagner, *The Craft of a Chinese Commentator*, p. 166). This project has developed into that of a research group in Heidelberg and was the topic of a conference 'Text and Commmentary in Imperial China', held in Heidelberg, in June 2000.

history. What is at stake, in this respect, with *The Nine Chapters* is to consider how our understanding of these matters could benefit from considering the case of mathematical Canons. More generally, virtually all scholarly disciplines identified some texts as being Canons, and research on this type of text has so far hardly begun to take into account Canons in scholarly disciplines. This article is an invitation to fill this gap. The key issue, in my view, is hence to determine how considering the Canons elaborated within the context of specific fields of knowledge such as mathematics can improve our understanding of the phenomenon of canonicity in general. Could it be that, in these specific contexts, we are in a better position to describe modes of reading or to account for attitudes and expectations? In this article, I'll attempt to illustrate why I believe so.

The second issue connects with the history of mathematics more specifically. Being essentially composed of problems and procedures of computation for solving them, *The Nine Chapters* has been read, quite anachronistically in my view, as a collection of recipes or a kind of textbook for primary school.[3] This approach, common among contemporary historians of mathematics, can hardly account for the properties Chinese readers attributed to the book in the past. This should sound as a warning that such an approach should be reconsidered. The questions raised above stem from the belief that our reading of *The Nine Chapters* can gain from situating the book more adequately within the category of Canons.

With these two agenda in mind, in a first part, I shall introduce one recurring expectation towards *The Nine Chapters* that appears to derive from the fact that it was approached as a Canon, namely, that at different moments in history, readers who left evidence of how they approached the book betray their assumption that it should encompass the whole of mathematics, which, in ancient China, essentially meant all mathematical procedures. Below, we shall examine more closely in three cases what form these statements of completeness took with respect to mathematics. How is it possible to believe that a collection of problems and procedures of computation could be all encompassing? How are

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[3] Compare Jean-Claude Martzloff, *Histoire des mathématiques chinoises*, (Paris: Masson, 1987), p. 116 (quotation by Zeng Guofan), p. 119 (quotation by Michael Loewe), p. 122, p. 124 (in the English translation, *A history of Chinese mathematics,* (Heidelberg: Springer), respectively, pp. 128, 131, 132-133, 134). It should be stressed that we have no evidence regarding the reasons why *The Nine Chapters* was compiled. The fact that seven centuries later, it was used as a textbook in the university does not tell us anything about its first uses. Nor does it exhaust the meanings readers attributed to the book.

we to understand what, at first sight, appears to be an extravagant claim? One way to deal with these questions is to consider that these statements are mere rhetoric, designed to enhance the status of the Canon. However, I believe that, before deciding on this issue and discarding these claims as intellectually insignificant, we should first at least try to understand them. The essential point for me is that, in the case of mathematics, one can argue for a precise interpretation of these claims. This is the aim of the second part of the article. With respect to *The Nine Chapters*, I shall offer an interpretation of the expectation that the book encompasses all of mathematics. To this end, I shall describe modes of reading the text and practicing mathematics that seem to have been put into play in relation to this belief. This may help us understand this kind of statement more generally, as regards other Canons.

We can address such questions because our sources provide evidence enabling us to observe the habits of Chinese readers over quite a long time span. Indeed, as was usually the case for Canons, commentaries were composed on *The Nine Chapters*, some being chosen through tradition to be handed down together with the text of the Canon itself. These commentaries reveal attitudes and beliefs regarding the Canon. They display ways in which the scripture was interpreted and they reveal hypotheses underlying the exegesis. The commentaries will provide the source materials that will serve as the basis for tackling our questions. In fact, no extant edition of *The Nine Chapters* survived that did not include the commentaries completed by Liu Hui in 263 and those composed by a group under Li Chunfeng's supervision and presented to the throne in 656. I shall designate the latter as "Li Chunfeng's comments." Incidentally, the first known source that refers to *The Nine Chapters* as a "Canon (*jing*)" is Liu Hui's preface to his own commentary.[4]

[4] Qian Baocong, *Suanjing shishu. Qian Baocong jiaodian* (Critical punctuated edition of *The Ten Canons of Mathematics*), 2 vols (Beijing: Zhonghua Shuju, 1963), vol. 1, p. 91. Chemla & Guo, *Les Neuf chapitres*, pp. 126–129. For the text of *The Nine Chapters* and its earliest commentaries, in addition to referring to our critical edition and translation into French published in the latter, I shall refer systematically to the critical edition contained in Qian, *Suanjing shishu*, vol. 1, since it presents the Canon within the context of *The Ten Canons*, that is, within the context in which the Li Chunfeng's commentary was composed. The interpretation of the sentence in question, which is translated in excerpt (1) below, can be contested. In order not to overload the main text with technicalities, I discuss in Appendix 1 the evidence regarding this point, on which my conclusions disagree with those of present-day historiography.

Several centuries appear to have passed with nothing known to have been written about *The Nine Chapters* until a revival of interest during the Song dynasty. After what the available evidence shows as an interruption. In 1084, the Department of the Imperial Library (*bishu sheng*) printed an edition of the Canon with its two traditional commentaries; the reprint of this edition by Bao Huanzhi around 1213 is the earliest extant edition known today.5 In relation to this, new commentaries were composed, such as Jia Xian's *Detailed Procedures of Huangdi's Canon of the Nine Chapters on Mathematics* (*Huangdi jiuzhang suanjing xicao*), in the first half of the eleventh century, printed by Rong Qi in 1148, and Yang Hui's *Detailed Explanations of The Nine Chapters on Mathematical Methods (Xiangjie jiuzhang suanfa*) in 1261. Through these editions and commentaries, over a time span of almost 1000 years, the historian can thus observe specific readers who made explicit their expectations towards, and their interpretation of, *The Nine Chapters*. Let us hence start by observing some features of their approach to the Canon.

THE COMMENTATORS' EXPECTATIONS OF THE CANON

When observing the attitudes towards *The Nine Chapters* manifested by the commentators, the striking fact one is confronted with is that, although Liu Hui, Li Chunfeng and Yang Hui operated at very different time periods and although their commentaries present important differences, they share some surprising expectations towards the book that seem to relate to its status as a Canon. To bring this point to light, let us concentrate on declarations made by each of them. If, at first sight, they appear somewhat obscure, they should become progressively clearer as I develop an interpretation of the key

[5] These editions of *The Nine Chapters* were produced within the context of larger enterprises for editing *The Ten Canons of Mathematics*. Bao Huanzhi apparently added postfaces to each of the Canons, four of which are, to my knowledge, still extant. Two of them, including that to *The Nine Chapters*, are known thanks to later sources, and are dated from the same date of 1200. Two others, whose original printings still exist, are signed 1212 and 1213. This seems to indicate that the editorial enterprise lasted several years, and was probably completed soon after 1213. This also indicates that *The Nine chapters* was among the first of all the Canons to be prepared for this edition. I am glad to be able to express my thanks to Professor Huang Yi-Long, who helped me clarify this issue. On Bao Huanzhi's edition, see K. Chemla,"Commentaries: The progressive "discovery" of a neglected source and of its genre", in F. Bretelle-Establet (éd.), *Looking at it from Asia: The Processes That Shaped the Sources of History of Science*, forthcoming.

elements that they introduce and that are all essential for my argument. In this first part, therefore, I suggest that the reader concentrate on the aspects of the declarations related to the completeness of *The Nine Chapters*. The other elements of the quotations will be progressively addressed and clarified as the argument unfolds.

Liu Hui's conception of mathematics

The first declaration, by Liu Hui, occurs in his preface to his commentary, when he describes how he conceives of the genesis of the text of *The Nine Chapters* and his own commentary. He writes:[6]

> (1) (...) Records tell that Li Shou created mathematics[7] (...). It is only when the Duke of Zhou established the Rites that [we know that] the nine parts of mathematics existed. The development (*liu*) of these nine parts, this is precisely what *The Nine Chapters* is.[8] Formerly, the cruel Qin burnt the books.[9] The procedures of the Canon

[6] See Qian, *Suanjing shishu*, pp. 91–2; Chemla & Guo, *Les Neuf chapitres*, pp. 126–9. In the following footnotes, I shall give only sketchy indications to explain some of the main features of the preface. For many points or statements that cannot be developed here, I refer the reader to my footnotes to the preface as well as to my glossary of mathematical terms published in Chemla & Guo, *Les Neuf chapitres*. In the paper, I shall number the passages quoted so as to refer the reader back to them more conveniently. Note that, in the translations given below, I use squared brackets to designate elements that are not explicit in the text, but that express my interpretation of the text. By contrast, round brackets insert transcriptions of the Chinese terms translated into pinyin or "explanations."

[7] Li Shou is said by late sources to have been a minister to the first Emperor Huangdi, a figure of legend, and to have created mathematics. When in the next quoted sentence, Liu Hui mentions the Duke of Zhou and the differentiation of mathematics into nine branches, he speaks of known history and the foundation of Zhou dynasty in the eleventh century BCE.

[8] The idea that the book derives from a "development" recurs with the following commentator, Li Chunfeng, in the seventh century; see below. On this part of the translation, see footnote 13, in Chemla & Guo, *Les Neuf chapitres*, p. 752.

[9] Liu Hui attributes the interruption of what had been, in his view, until that point, a smooth transmission of the text of *The Nine Chapters* to the burning of books ordered by the Emperor Qin shi huangdi who unified the Chinese empire in 221 BCE. On the next sentence, see the discussion in Appendix 1.

got scattered and damaged. After that time, the Bei Ping Marquis Zhang Cang[10] and the Assistant of the Grand Minister of Agriculture, Geng Shouchang,[11] both acquired a universal reputation for their excellence in mathematics. On the basis of scraps of the old text (*wen*) that were handed down, Zhang Cang and others made both excisions (*shan*)[12] and completions. This is why, when one examines its sections, in places they differ from the ancient ones and what is discussed is much in modern terms.

As a child, I studied *The Nine Chapters;* as an adult, I again looked at it in detail. I observed the dividing of *Yin* and *Yang*, synthesized the source of mathematical procedures. Having spent much time to fathom its depths, I managed to understand its meaning/intention (*yi*).[13] This is why I dared (...) compose a commentary on it.

The accomplishments (*shi*) and their categories develop in relation to one another, but they each have that to which they return/amount (*gui*).[14] Therefore, the reason

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[10] Zhang Cang was a civil servant who held quite important positions in the administration of the Han Empire from its beginnings. His domains of expertise included mathematics, astronomy, bookkeeping, management of finance and exegesis of some Confucian Canons. See Guo Shuchun, chapter B in K. Chemla & Guo S., *Les Neuf chapitres*.

[11] Like Zhang Cang, but later, in the first century BCE, Geng Shouchang was a civil servant who contributed to the administration of finance and to settling astronomical questions. See Guo Shuchun, chapter B in K. Chemla & Guo S., *Les Neuf chapitres*.

[12] This operation of expurgating inauthentic material that had accumulated in the documents during the process of transmission characterizes the way in which the tradition from at least the Han dynasty onwards conceived of Confucius as editor of Canons; see Henderson, *Scripture, Canon and Commentary*, pp. 26ff. In the first piece of literary criticism devoted to Canons (*Wenxin diaolong,* chapter "Revering the Canons (*Zong jing*)"), at the turn of the sixth century, Liu Xie refers to Confucius's editing with the same word: *shan*. On this point, see Stephen Owen, *Readings in Chinese Literary Thought*, (Cambridge, Mass.: Council on East Asian Studies, Harvard University Press, 1992), pp. 194–5.

[13] As is suggested by the development of this term below, this may also be understood in the plural: "its meanings/intentions." Alternatively, if the anaphora refers to the mathematical procedures, this can also be interpreted as "their meaning/intentions." We will come back to this point.

[14] One can also interpret this as "The categories of the accomplishments develop in relation to each other." In this context, "accomplishment" probably refers to the mathematical problems contained in *The Nine Chapters*

and the procedures developed in relation to the problems, as well as refers to mathematical objects (On the relationship between objects and procedures in ancient China, see Karine Chemla, "Nombre et opération, chaîne et trame du réel mathématique. Remarques sur le commentaire de Liu Hui aux *Neuf chapitres sur les procédures mathématiques*," in Alexei Volkov (ed.), *Sous les nombres, le monde. Extrême-Orient, Extrême-Occident,* 1994, 16:43–70.) This passage, essential for delivering Liu Hui's conception of mathematical reality, requires a delicate interpretation. Here is, in my view, a plausible interpretation on the basis of other occurrences of the key terms in his commentary (*lei* "category/class," *shi* "accomplishments," *gui* "return/amount to"; for all these terms, I refer the reader to my glossary in K. Chemla & Guo S., *Les Neuf chapitres*). With respect to objects, fractions provide a good illustration of how to understand the passage, if one relies on the commentary on the procedure for adding up fractions (*hefenshu*, after problem I.9). A fraction can be given in two ways: a set of parts (two of three parts for 2/3) or a pair of numbers—numerator and denominator. Basically, a fraction represents a quantity that is expressed on the basis of its structuration into parts. Stating that fractional quantities belong to the same class or category implies that they share the same denominator or are structured with parts of the same size. A given fractional quantity can change category by transformation. Such is the case when one systematically cuts its parts into finer parts (four of six parts, instead of two of three parts), or conversely gathers its parts into coarser ones (two of three parts, instead of four of six parts), or when, correspondingly, one multiplies (2/3 becoming 4/6) or divides (4/6 becoming 2/3) its numerator and denominator by a same number. Expressing the fractional quantity requires that a particular category be adopted. The possibility of transforming fractions and thereby changing their categories is the essential property that allows computing with them and bringing them together, since computing requires that categories of the various fractions involved in the operation be transformed into the same category. In this sense, "the categories of the realizations (or the realizations and their categories) develop with respect to each other." However, as quantities, all previous fractions return/amount to the same. Such is, in my view, the set of facts covered by the passage, when considered with respect to fractions. A similar interpretation can be developed for geometrical bodies on the basis of Li Chunfeng's commentary on the extraction of the spherical root (*kailiyuanshu*, after problem 4.24). Moreover, mathematical problems are also understood as having categories and forming classes defined on the basis of the procedures attached to them. Rephrasing a procedure may change the category, the class to which the problem belongs, with each formulation revealing connections with other problems from a given perspective. We shall analyze below examples of such connections and the mathematical practices attached to the reformulation of procedures. However, beyond reformulation, procedures performing the same task or evaluating the same entities, "amount/return to the same" (see Liu Hui's commentary after the "procedure for multiplying parts," problem 1.21). Moreover, as will become clear in what follows, through reformulation, unification can be carried out between procedures that appeared at first sight to be distinct. They are thereby revealed as having derived from the same root before they differentiated into distinct computations. This is why Liu Hui can speak of a "source" or a "stem" for mathematical procedures, out of which the procedures develop in distinct categories, relative to each other. To sum up, all mathematical realities can be transformed, according to modalities that need to be analyzed, in relation to the why, although they divide into branches, they share the same stem is that they emerge from only one of their ends (*duan*). Furthermore, if the internal constitutions (*li*)[15] are analyzed with statements (*ci*) and if the bodies are dissected with figures (*tu*), [one sees that *The Nine Chapters*, as restored during the Han dynasty,] gets close to, though made simple, being able to encompass and, though bringing into communication, not being confusing. (...) Although one speaks of "the nine parts of mathematics," they have the capacity to exhaust the subtle (*xian*) and to penetrate the minute (*wei*), to fathom what knows no bounds (what has no location—the *shen*). (My emphasis)

Let us, at this point, leave aside several aspects of this difficult text to concentrate on only some remarks. Note, first, that Liu Hui considers *The Nine Chapters* from the perspective of the ability of the Canon to encompass mathematics (they "get close to, though made simple, being able to encompass..."). More generally, the commentator stresses the unlimited potentiality of the "nine parts of mathematics," which, in his view, developed into *The Nine Chapters*, contrasting it with the moderation of its size. If one may note a nuance in Liu Hui's assertion of the comprehensiveness of the Canon ("they get close to being able to encompass"), in my view, it is directed at Zhang Cang's and the other Han editors' inability to adequately restore the Canon, which had been. damaged during the Qin burning of books.[16] This interpretation is supported by the second part of Liu Hui's preface, where, after having introduced a problem and evoked some procedures of *The Nine Chapters*, he states:

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realities with which they interact, or to those with which they are compared or can be unified, and hence they can change category (*lei*). These transformations are deemed essential to the practice of mathematics, in that they bring different realities into communication or unify them. However, throughout these transformations, that "to which they amount" (*suogui*, in the next part of the sentence) does not change. The concepts *lei* and *suo gui* form, I believe, a contrasting pair.

[15] See my glossary in K. Chemla & Guo S., *Les Neuf chapitres* and below.

[16] On the idea that the preface describes the compilation of *The Nine Chapters* as a process of recovery of a lost scripture, see K. Chemla, "Antiquity in the shape of a Canon. Views on antiquity from the outlook of mathematics," in Dieter Kuhn (ed.), *Monumenta serica Monograph Series*, forthcoming.

(2) But [in *The Nine Chapters*] there is nothing of the category (*lei*) of (…I skip the problem Liu Hui introduced here...).[17] Therefore the procedures made by Zhang Cang and the others do not yet suffice to exhaust extensively all mathematics (*bo jin qun shu*).[18] Within the nine parts of mathematics, I investigated the one named "double difference". I examined (*yuan*) its essential points (*zhiqu*) so as to make them extend to/be efficient for (*shi*) this [problem].

(…The statement of procedures allowing problems of this class to be solved follows…) I elaborated the "double difference" and wrote a commentary on it so as to explore the meaning/intention (*yi*) of the ancients. I joined it after [the chapter] "base and height " (*gougu,* i.e., the last chapter). (My emphasis)

As one can see here, again, the expectation made explicit is that the restored Canon should "exhaust all mathematics." The blame for the failure of *The Nine Chapters* to do so is put on its Han editors, including Zhang Cang, and not on the Classic itself. This interpretation is, in its turn, supported immediately afterwards by the way in which Liu Hui expounds his method for dealing with what he feels is a lacuna that must be filled. To this end, he does not claim to have invented the procedures he introduces in his preface. On the contrary, in a way that recalls his description of how the Han editors proceeded, he shows how he obtained these procedures through "investigating," in a certain way, one procedure found in "the nine parts of mathematics." He therefore maintains that, for the purpose of filling the gaps in *The Nine Chapters*, he is using what he stated in (1) to be its source: the "nine parts of mathematics." It is interesting to note, in this respect, that Liu Hui's last

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[17] The category of a problem is defined by a procedure solving it, see Karine Chemla, "Qu'est-ce qu'un problème dans la tradition mathématique de la Chine ancienne ? Quelques indices glanés dans les commentaires rédigés entre le 3[ième] et le 7[ième] siècles au classique Han *Les Neuf chapitres sur les procédures mathématiques,*" *Extrême-Orient, Extrême-Occident*, 1997, 19:91–126. As is clear from the next statement, Liu Hui states here that the Canon as restored during the Han dynasty fails to provide the means to solve the given problem.

[18] Evidence that *shu* can refer to mathematics or the parts composing mathematics is provided in my glossary, see entry on *shu*, Chemla & Guo, *Les Neuf chapitres*, pp. 984–986.

sentence in (2) seems to indicate that he conceives of himself as continuing the work of editing the text of the Canon, based on the pieces of scripture available and guided by the assumption that *The Nine Chapters* should be complete. If such is the case, then we have here a concrete description of the editorial procedure, as practiced by Liu Hui, that indicates the mathematical activities involved in editing. We shall come back below to his account of how he relies on a given procedure to derive means to fill up the lacunae.

Other elements in these excerpts are worth stressing. According to my interpretation of the former quotation (1), the commentary seems to be conceived as the exercise bringing to light the capacity of the Canon to fully encompass mathematics. More precisely, it is the analysis of the "internal constitutions," or *li*'s, and the dissections of the bodies that bring to light the properties of the Canon that are emphasized, i.e., that it "get close to, though made simple, being able to encompass."

In addition to this, Liu Hui makes explicit that what prompted the writing of his commentary was having reached an understanding of the "meaning(s)" or the "intention(s)" (*yi*) of *The Nine Chapters*, or of its procedures. In the same vein, in the latter passage (2), he again describes his commentary as aiming at "exploring the meaning (*yi*)" of the ancients. One may be tempted to interpret this term quite loosely. However, since understanding the *yi* appears to be an essential goal in his exegesis, I suggest, at least in the beginning, not jumping to a lax conclusion, but rather attempting to determine whether this term might not refer to something more specific.[19] We shall come back to interpreting these terms below.

Lastly, let us stress that, in the same lines of his preface where he introduces the theme of the completeness of the Canon, Liu Hui also alludes to an architecture of

[19] The fact that we are dealing with a text devoted to mathematics will prove here to be quite helpful in this respect. In fact, I argue below that, even though Liu Hui uses here quite a general term, in the context of mathematics, *yi* takes up a more precise denotation that I shall discuss. The way in which the term is used in the commentary shows that the nature of the *yi* Liu Hui is after takes a specific form. If this is the case, it is easy to understand why the study of Canons produced by the scholarly disciplines could contribute to the study of canonicity in general. This does not mean, however, that this "meaning" would be unique and straightforward, as is demonstrated by the fact that the task of exegesis went on for centuries in China.

mathematical "accomplishments" as deriving, by a process of differentiation, from a unique stem.[20]

To sum up, two features emerge from this declaration that will prove essential for us. First, the Canon is expected to be all encompassing. Secondly, a structuring of mathematical procedures is sketched, in which various mathematical realities differentiated from a common stem.

Li Chunfeng's approach to the completeness of the Canon

Interestingly enough, a comparable combination of elements is to be found in a declaration by the seventh-century commentator Li Chunfeng, who also reveals his belief that *The Nine Chapters* somehow encompassed all mathematical procedures. This passage contains several components that will be elucidated below, but first, let us quote it to compare its overall idea with what we just saw. This declaration, included in the "Monograph on the musical scale and the calendar *(Lülizhi*)" of the *History of the Sui Dynasty* (*Suishu*) that was prepared under Li Chunfeng's supervision,[21] reads as follows:

> (3) As for what is called *lü*,[22] there are <u>nine [parts of mathematics] that flow from them</u>: the first is called "rectangular fields" (…Li Chunfeng lists here the titles of all

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[20] Guo Shuchun, *Gudai shijie shuxue taidou Liu Hui (Liu Hui, a Leading Figure of Ancient World Mathematics*), (Jinan: Shandong kexue jishu chubanshe, 1992), chapter 8, pp. 301–320, is devoted to analyzing the structure of mathematics meant by Liu Hui. His results are synthesized on p. 318.

[21] Yang Jialuo, *Zhongguo tianwen lifa shiliao* (Historical Material on Chinese Astronomy and Calendar), (Taipei: Dingwen shuju, 1978), vol. 3, p. 1859.

[22] This term, which I leave untranslated, designates quantities that are defined only with respect to each other. *The Nine Chapters* introduces the concept for describing the rule of three. The quantities that govern how to exchange the considered entities are designated as *lü*. The terms of a ratio can be designated as *lü* and may, as such, represent any entities forming a proportion with them. If, for instance, one considers that the diameter of a circle is to its circumference as 1 to 3, the quantities are said to form, respectively, a *lü* of the diameter and a *lü* of the circumference. The commentators qualify as *lü* a greater variety of such sets of quantities, such as, for instance, the coefficients of a linear equation. For greater detail, see Li Jimin, *Dongfang shuxue dianji* Jiuzhang suanshu *ji qi Liu Hui zhu yanjiu (Research on the Oriental Mathematical Classic* The Nine Chapters on Mathematical Procedures *and on its Commentary by Liu Hui)*, (Xi'an: Shaanxi renmin jiaoyu chubanshe, 1990), pp. 136–161, or my glossary, in K. Chemla & Guo S., *Les Neuf chapitres*.

chapters of *The Nine Chapters* and Liu Hui's commentary on his interpretation of the main purpose of these chapters)…. They (i.e., the nine parts of mathematics) all multiply to disaggregrate them (i.e., *lü*'s), divide to assemble them, homogenize and equalize to make them communicate, apply the [procedure] of "suppose" (i.e., the rule of three) to link them together, hence the methods of the mathematical procedures are exhausted by these. (My emphasis)

Li Chunfeng thus also both asserts that the Canon is complete in a sense ("...the methods of the mathematical procedures are exhausted...") and describes an architecture of mathematical knowledge, essentially revolving around mathematical procedures. The articulation between these two elements is worth stressing. The statement regarding the comprehensiveness of the Canon appears as a *conclusion* and is based on the architecture described. The reasoning underlying the assertion can be sketched as follows: the nine chapters of the Canon have a common origin—*lü*, a concept to which we shall come back—and their procedures all make use of the same and limited set of fundamental operations applied to such entities. If, from this notation, Li Chunfeng can *deduce* that, more generally, these also "exhaust the methods of the mathematical procedures," this implies that he assumes that the property established for *The Nine Chapters* extends to the whole of mathematics. This deduction hence presupposes a precise conception of *how* the Canon encompasses all mathematical procedures. This articulation, which is quite concrete and elaborate, apparently supports our assumption that the belief in the comprehensiveness of the Canon is not vague and superficial, but that we should be able to account for how the Canon could be perceived as all-encompassing. This indicates the direction we will pursue.

Yang Hui's perception of the Canon

In addition, statements from the Song and Yuan dynasties (960-1368) are clear-cut in expressing the same expectation. For instance, the thirteenth-century commentator Yang Hui asserts:

(4) Everyone who studies mathematics (*suan*) considers the method for multiplication as capital. Every time one puts [the terms] of the method [on the counting surface], one wants its result to be appropriate; if one determines the

> positions, so that one makes the numbers correspond to each other adequately, one wants it to be not false. In the case when, by division, one does not exhaust (the dividend), one takes the divisor as denominator and the dividend (i.e., that which, in the end, remains in the position of the dividend) as numerator (i.e., as becomes clearer below, division is considered from the point of view of what makes it the exact inverse of multiplication). If they are too complex (i.e., if numerator and denominator have a common divisor), one simplifies them; if, in return, one makes the parts communicate (i.e., one multiplies the integral part of the quotient by the denominator and adds the result to the numerator), then one returns to the origin (*huanyuan*). Such are the fundamental tools of multiplication and division (…A list of fundamental situations and operations contained in each of *The Nine Chapters* follows,[23] which all amount to diverse uses of multiplication and division…). These are what exhausts the inner constitution (*li*) of mathematical methods. *The Nine Chapters by Huangdi* is complete and subtle; it encompasses all situations (*qing*). It was undeniably written by a Sage. (My emphasis)

It is remarkable that this declaration presents a structure quite similar to Li Chunfeng's and includes similar elements to establish the exhaustiveness of *The Nine Chapters*. An architecture of mathematical knowledge is unfolded from a base consisting of multiplication and division. From this base, a list of fundamental operations that are the essence of the various chapters of the Canon is derived. This leads Yang Hui to assert the completeness of the Canon. The emphasis placed by Yang Hui on the "inner constitution of mathematical methods" echoes Liu Hui's preface. In addition, as in Li Chunfeng's case, Yang Hui's conclusion proceeds from identifying the fundamental operations at play in the procedures of the Canon to asserting that they "exhaust the inner constitution (*li*) of mathematical methods." Hence, the way in which the completeness of the Canon is approached is quite similar, which confirms that this is the direction to be explored to

[23] I keep for another paper a detailed analysis of this architecture of mathematical operations. In relation to his commentary, Yang Hui produces a reclassification of the procedures of the Canon, which ought to be confronted with the architecture he outlines in his preface.

account for this belief. The recurrence of the same arguments makes it all the more difficult to discard such statements as less than meaningful.

Recapitulating

In conclusion, even if their declarations display differences, the three commentators concur in believing that *The Nine Chapters* does, or should, encompass all mathematical procedures. In contrast to Li Chunfeng and Yang Hui, who both assert the completeness without restraint, Liu Hui's statement is less affirmative. However, as I have argued, his restriction seemingly regards the text of the Canon as having been edited by previous generations and, in his view, the genuine Canon should be comprehensive. Hence, the nature of the expectation remains the same.

As a side remark, note that, if the commentators believed *The Nine Chapters* encompassed mathematical reality, it is no wonder that their mathematical activity would be to compose a commentary on the Canon.

More importantly, the essential point here is that, as John Henderson (1991, pp. 100 ff.) has shown, this feature constitutes a very common expectation that Chinese commentators regularly exhibited with respect to Canons. One early expression of this belief can be found in the canonical corpus itself, for instance in the "Great Commentary" (*Xici dazhuan*), which states with respect to the *Book of Changes*: "The *Changes* (*Yi*) is broad, great and all-encompassing. There are in it the Way of heaven, the Way of man, and the Way of earth."[24] The corresponding expectation encountered with respect to *The Nine Chapters* therefore probably only expresses that the book was perceived as a Canon, comparable to, among others, the Confucian ones. We do have evidence that such a comparison was made by some actors themselves. One of the clearest such expressions can be found in a passage by the thirteenth-century commentator, Yang Hui, who is himself quoting the preface composed by Rong Qi when he had Jia Xian's (eleventh-century) commentary printed in 1148. Yang Hui states: "When the government instituted the examinations in mathematics to select officials, they chose *The Nine Chapters* to be the most important of the mathematical Canons, since, indeed, it is like the six Canons of the Confucians, the (*Canon of) Difficulties* and the *(Grand) Simplicity* of the medical schools,

[24] *Xici*, *xia,* chap. 10. Trans. quoted in Henderson, *Scripture, Canon and Commentary*, p. 101.

the *Book of Master Sun* of military art!" This piece of evidence, incidentally, legitimates the idea that we may study the attitudes towards *The Nine Chapters* to approach the phenomenon of canonicity more broadly.[25]

In other words, placing *The Nine Chapters* against the backdrop of other Canons provides a context for the expectation that it be all encompassing. This does not, however, yield an interpretation for this expectation. This is what we can set out to do, for the case of mathematics, by relying on the three assertions made by our commentators.

A second point, common to our three commentators, deserves some attention: they all describe an architecture of mathematical reality in connection to stating the comprehensiveness of the Canon. The commentators also display different conceptions of the architecture of mathematical procedures and of how *The Nine Chapters* encompasses mathematics. Liu Hui's assertion that mathematical "accomplishments" "share the same stem" may be interpreted as expressing views that are quite close to Li Chunfeng's. Indeed, in his commentary on the addition of fractions (*hefenshu*),[26] Liu Hui states:

> (5) Multiply to disaggregate them, simplify to assemble them, homogenize and equalize to make them communicate, how could those not be the key points (*gangji*) of computations/mathematics (*suan*)?

Without yet discussing the meaning of this assertion, one can notice that it focuses on three of the four fundamental operations that lie at the core of Li Chunfeng's declaration quoted above (3). These operations, according to Liu Hui's description of the architecture of mathematics, may therefore relate to the stem from which the procedures "diverge into branches." In fact, when considering his entire commentary, this is indeed the most plausible

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[25] Nathan Sivin, "Text and experience in classical Chinese medicine", in Don Bates (ed.), *Knowledge and the Scholarly Medical Traditions*, (Cambridge, UK: Cambridge University Press, 1995), pp. 177–204, on pp. 191–2, alludes to the fact that the medical Canons were also considered by some of their readers as all-encompassing. We hope that further research may cast light on the ways in which their exegetes interpreted the medical Canons to reach this conclusion. This, too, would allow the phenomenon of canonicity to be dealt with from a comparative perspective.

[26] This passage, mentioned earlier, can be read after problem 9 of chapter 1. See Karine Chemla, "Theoretical aspects of the Chinese algorithmic tradition (first to third century)," *Historia Scientiarum,* 1991, 42:75–98 and errata in the following issue.

interpretation.[27] As a result, the architectures described by our two commentators would partially coincide. Two main differences can be noticed, though. First, Li Chunfeng diagnoses that a fourth fundamental operation, the "rule of three," should be added to the list of key points suggested by Liu Hui. Moreover, the architecture described by the seventh-century commentator not only consists of a list of operations to which all procedures can be reduced, but it also involves a source—the concept of *lü*—from which *The Nine Chapters* develops and to which the key operations are applied. Yet, beyond these differences, both architectures share fundamental features.

In contrast to the other commentators, Yang Hui (4) claims that a pair of opposing but complementary operations, namely, multiplication and division, is at the basis of mathematics. From this pair, the other fundamental operations at play in the various chapters of the Canon are derived, which then serve to encompass all mathematical procedures, in a sense still to be made clear.

However, despite the divergences, it remains common to our three commentators that their formulation of the expectation of completeness always went along with making explicit a conception of an architecture of mathematical procedures. Furthermore, the various architectures described share the feature of displaying how mathematics develops from a limited list of fundamental operations and how these can be exhibited by commenting on the Canon.

These facts raise several questions relating to "Canons" as a type of text. First of all, how are we to interpret that a book may be all encompassing? I hope, at this point, to have convinced the reader that there are reasons to persist in attempting to understand this assertion, however strange it may seem. The whole purpose of the article is to offer an interpretation for the case of mathematics. We leave it to further research on Canons to assess how general this interpretation may be. Another related question also requires some elucidation. In our case, more specifically, how does the belief in the all-encompassing nature of the Canon relate to providing a description of the structure of mathematics that

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[27] Perhaps, the procedure in the comment of which Liu Hui brings to light the operations listed in (5) —the addition of fractions—could then be the unique "end" from which mathematical realities are said to emerge. But there is no evidence I can think of that would confirm this hypothesis. Guo Shuchun, *Gudai shijie shuxue taidou Liu Hui*, especially pp. 315 *ff*, offers a different interpretation, placing the notion of measure at the root of the tree of mathematics.

shares the features outlined above? Dealing with the issues will lead us to the main questions of this essay: How was such a kind of book as a Canon read? How does the kind of exegesis carried out by our commentators connect with these issues? In what follows, I shall address these questions briefly, relying mainly on the commentary ascribed to Liu Hui.

MODES OF READING THE CANON

A glimpse at the canon

As already noted, the expectation that a book is all encompassing, that it contains all mathematical procedures, may disconcert a modern reader. This astonishment may become even deeper when skimming through *The Nine Chapters*. In fact, the Canon is composed of 246 particular problems, their numerical answers and algorithms that solve them. Let us quote some examples of these, to give an idea of the whole.

The ninth chapter, "Base and Height" (*gougu*), which is devoted to the right triangle, opens as follows:[28]

> (6) SUPPOSE THAT THE BASE (*GOU*) IS WORTH 3 *CHI* AND THE HEIGHT (*GU*) 4 *CHI*. ONE ASKS HOW MUCH THE HYPOTENUSE MAKES.
>
> ANSWER: 5 *CHI*
>
> (…[29])
>
> PROCEDURE OF THE BASE AND THE HEIGHT:
>
> BASE AND HEIGHT BEING EACH MULTIPLIED BY ITSELF, ONE ADDS (THE RESULTS) AND DIVIDES THIS BY SQUARE ROOT EXTRACTION, WHICH GIVES THE HYPOTENUSE

[28] See Qian Baocong, *Suanjing shishu*, vol. 1, p. 241, Chemla & Guo, *Les Neuf chapitres,* pp. 704–5, problem 1 of chapter 9. Throughout the paper, I quote the Canon in capital letters so as to distinguish its text from that of the commentaries.

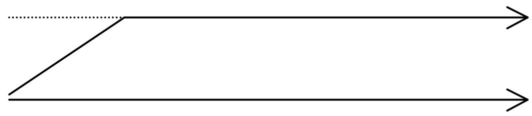
[29] This first problem is followed by two other ones, deduced by applying a circular permutation to the data and the unknown. Moreover, the first procedure, quoted at the end of (6), is followed by similar procedures allowing the third side of a right triangle to be determined given any two sides. I skip these other problems and procedures here.

We recognize here, in the form of a procedure, what, in ancient China, corresponded to the "Pythagorean theorem." This problem is specific as regards its data, but the situation it involves is abstract. However, most of the other problems are specific in both respects, as is illustrated by the following example:[30]

(7) SUPPOSE THAT A GOOD WALKER WALKS 100 *BU* WHILE A BAD WALKER WALKS 60 *BU*. IF, NOW, THE BAD WALKER FIRST WALKS 100 *BU*, BEFORE THE GOOD WALKER [STARTS] PURSUING HIM, ONE ASKS IN HOW MANY *BU* HE WILL CATCH HIM UP.

ANSWER: 250 *BU*

100 *bu* Distance walked by the bad walker



Distance walked by the good walker[31]

The algorithm given to solve this problem is expressed in terms relating to the described situation and uses concrete numbers:[32]

(7B) PROCEDURE: ONE PLACES (ON THE COUNTING SURFACE) THE 100 *BU* OF THE GOOD WALKER, AND SUBTRACTS FROM IT THE 60 *BU* OF THE BAD WALKER; THERE REMAINS 40 *BU*, WHICH IS TAKEN AS DIVISOR. ONE MULTIPLIES, BY THE 100 *BU* OF THE GOOD WALKER, THE 100 *BU* THAT THE BAD WALKER HAD FIRST WALKED, WHICH MAKES THE DIVIDEND. DIVIDING THE DIVIDEND BY THE DIVISOR GIVES THE RESULT IN *BU*.

Understanding how 246 such problems and procedures for solving them could be conceived of as encompassing all mathematical procedures is the challenge confronting us.

Observing how commentators read *The Nine Chapters* and how they approached mathematics will provide some clues. This implies that, if we do not take these precautions,

[30] See Qian Baocong, *Suanjing shishu*, vol. 1, pp. 191–192, Chemla & Guo, *Les Neuf chapitres*, pp. 517–519, problem 12 of chapter 6.

[31] I add this diagram and the following to help the reader. They do not belong to the original text.

[32] We come back to its correctness below.

and if, as modern readers, we anachronistically read the Canon as a set of problems and algorithms, we run the danger of completely missing the significance of *The Nine Chapters* for its Chinese readers of the past. Let me stress again that this is precisely what is at stake in this approach, with respect to the history of mathematics.

The commentaries attest to several distinct modes of reading that were put into play and that, each in a different way, further developed the text of *The Nine Chapters* beyond its simple appearance. I have already dealt with some of them elsewhere, and I shall simply summarize some results here.

Reading a problem

A first mode of reading illustrates how one item can be taken to stand for a multitude. Indeed, it can be shown that, as one may expect, the commentators did not read a problem as a particular case, but as a paradigm, in the grammatical sense of the word, that stood for a whole category of problems. Interestingly enough, Liu Hui explicitly relates this reading of problems to the way in which, according to the *Analects,* Confucius expects his disciples to develop his teachings: "The Master said: (...) If I hold up a corner and a man cannot come back to me with the other three, I do not continue the lesson."[33] Accordingly, Liu Hui refers to describing a method within the context of a problem as "holding up a corner."

In our case, it can be shown more precisely that it is the procedure given to solve a problem that provides the basis for determining the class of problems the given problem stands for.[34] The class consists of all the similar problems that one procedure can solve. It does happen, though, that the procedure given by *The Nine Chapters* is not as general as it could be, and hence that the problem does not stand for as large a class as it could. Liu Hui indicates such cases and reformulates a more general algorithm in his commentary. This reveals that the commentator expects that a procedure will be general and that a problem will

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[33] Arthur Waley, *The Analects of Confucius*, London, Unwin Hyman, 1988 (first published in 1938 by George Allen and Unwin), p. 124 (Book VII, § 8).

[34] See Chemla, "Qu'est-ce qu'un problème", pp. 96–7. Note that, in passage (2) quoted above, Liu Hui mentions looking for a problem in the Canon that would belong to the same category as the problem he wants to solve. He concludes that the procedures gathered by Zhang Cang and others do not suffice, which reveals the relation between the procedure and the category.

stand for a class.[35] This interest in the generality of a procedure, as we shall see, extends quite far. Moreover, such examples show that Liu Hui, seemingly, does not manifest any undue awe towards *The Nine Chapters,* despite its status, and, in this case, simply indicates its limitations. Furthermore, let us stress that Liu Hui does not seem to particularly value formulating an *abstract* procedure that would *apply* to various particular cases. Instead, he prizes the *generality* of an algorithm that *circulates* from problem to problem as far as possible. This fits with his reading the Canon as expressing the general in terms of the paradigmatic.

Reading the arrangement of problems in the Canon

Observing Liu Hui's commentary reveals a second technique that is put into play when reading the Canon. The commentator appears to read meanings into the way in which the sequences of problems were organized within *The Nine Chapters*. This can be argued in several ways with regard to Chapter 8 "Measures in Square" (*Fangcheng*), which deals with systems of n linear equations with n unknowns. Liu Hui's commentary progressively extends the range of systems to which the basic algorithm can be applied by filling the gaps between the successive problems of the Canon. He hence appears to "interpret" the arrangement of the sequence of problems. Moreover, in this case, his interpretation of Chapter 8 can be shown to develop from understanding the terms of the first problem in two distinct ways, a concrete interpretation and a formal one.[36]

Chapter 9 also provides an example where Liu Hui appears to read a meaning in the arrangement of the problems. We mentioned above the first and abstract problem (6) with which this chapter devoted to the right triangle opens. The rest of the chapter consists of a sequence of seemingly more concrete problems. Liu Hui comments on the title "Procedure of the Base and the Height" as follows:

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[35] See Karine Chemla, "Generality above abstraction. The general expressed in terms of the paradigmatic in ancient China's mathematics," *Science in Context*, 2003, 16: 413–458.

[36] See Karine Chemla, "Les problèmes comme champ d'interprétation des algorithmes dans *Les Neuf chapitres sur les procédures mathématiques* et leurs commentaires. De la résolution des systèmes d'équations linéaires", *Oriens-Occidens*, 2000, 3:189–234.

> (8) (…) (Base, height, hypotenuse and the procedure linking these terms) are about to be extended (*shi*) to all the algorithms, this is why this procedure is set out first so as to make their origin (*yuan*) appear. (My emphasis)

This comment is interesting in several respects. Let us stress several key points that reveal a third aspect of the commentator's way of reading the Canon. First, Liu Hui's point aims at *explaining* the position of the first set of problems and procedures in the chapter: he reads a meaning, an intention, in its being placed at the beginning of the chapter.

Secondly, the way in which this location in the chapter is justified is most interesting, and it takes us closer to essential aspects of the commentary. The "procedure of the base and the height," an algorithm amounting to what we call the Pythagorean theorem, with its terms that define the shape of the right triangle, are described as the "origin" of the subsequent algorithms in the chapter. This relates to the fact that, in his comments accounting for the correctness of the algorithms, Liu Hui brings to light that each of the procedures for solving a problem in Chapter 9 involves, in one way or another, the sides of right triangles and the "procedure of the base and the height." Therefore, a first connection appears between "going back to their origin," to the source of the procedures—a recurring theme in Liu's preface, where we read in (1): "I ... synthesized the source of mathematical procedures"—and proving the correctness of the procedures by revealing that a general algorithm and its terms have been put into play.

Moreover, the relationship of this fundamental procedure and the geometric terms attached to it and the subsequent procedures is described as an "extension," in the sense that the procedure's use extends to resolving the situations listed in the rest of the chapter. We recognize the interest in the circulation of procedures, or in generality, mentioned above. This remark by Liu Hui thus reveals a second connection between the fact that a procedure, with the terms it involves, is the "origin" of others and its capacity to "spread" widely. In Liu Hui's understanding, putting the "procedure of the base and the height" at the beginning of the chapter amounts to highlighting these facts.

It is interesting that a similar connection between "going upstream" towards the "essential points" or the "origin" of a procedure and being able to increase the efficiency of the essential points further can be noticed in Liu Hui's preface (2). We recall that when the commentator was describing how he investigated a given procedure to find out how to extend it to solve the remaining problems, he wrote: "I examined (*yuan*) its essential points (*zhiqu*) so as to make them extend to/be efficient for (*shi*) this [problem]."

At this point, if we recapitulate what we have observed, the assumption emerges that "synthesizing the source of mathematical procedures" and finding out their "essential points" may relate to exhibiting some general procedures underlying particular mathematical procedures of the Canon and playing a part in the account of the correctness.

Could this assumption be that to which Liu Hui refers, when, twice in his preface, he speaks of his commentary as offering an understanding of "the meaning(s)" or "the intention(s) (*yi*)" of the Ancients or the Canon? I believe it may be so for several reasons. And, in order to capture from yet another angle Liu Hui's ways of reading *The Nine Chapters*, I suggest turning now to what his own commentary tells us of his conception of the "*yi*" of the Canon, a term which seems central in his own representation of his approach to the text.

Elucidating the nature of the *yi* of the Canon

Let us start from an occurrence of the term that is highly revealing of both an assumption the commentator makes with respect to *The Nine Chapters* and the mathematical practice related to inquiring into the "meaning/intention (*yi*)" of the Canon or its procedures. The context is the procedure given for the "extraction of the spherical root," in which the diameter of a given spherical volume must be determined. The Canon reads as follows:[37]

(9) SUPPOSE AGAIN ONE HAS A NUMBER-PRODUCT (*JI*)[38] OF 1,644,866,437,500 *CHI*. ONE ASKS HOW MUCH THE DIAMETER OF THE SPHERE MAKES.

ANSWER: 14,300 *CHI*.

PROCEDURE FOR EXTRACTING THE SPHERICAL ROOT:

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[37] Qian Baocong, *Suanjing shishu*, vol. 1, pp. 155–6, Chemla & Guo, *Les Neuf chapitres*, pp. 378–381, problem 24 of chapter 4. I refer the reader to Donald Wagner, "Liu Hui and Tsu Keng-chih on the Volume of the Sphere," *Chinese Science,* 1978, 3:59–79, where the whole commentary is treated in detail.

[38] *Ji* designates a number that is considered to have been yielded by a multiplication.

ONE PUTS THE QUANTITY OF *CHI* OF THE NUMBER-PRODUCT, MULTIPLIES IT BY 16, AND DIVIDES IT BY 9. TO DIVIDE WHAT IS OBTAINED BY EXTRACTION OF THE CUBE ROOT GIVES THE DIAMETER OF THE SPHERE.

In the opening section of his commentary, Liu Hui formulates a reasoning that may have produced the procedure. Let us follow it, before drawing some conclusions. First, Liu Hui identifies that the procedure "probably (*gai*)" rests on taking the ratio of the circumference of a circle to its diameter as 3 to 1. These values are used throughout *The Nine Chapters*. Liu Hui qualifies them as *lü*—the very concept placed by Li Chunfeng, in (3), at the origin of mathematical procedures. By this designation, the commentator indicates that they can be multiplied or divided by a number without affecting the ability of the new pair of quantities to express the relationship between the circumference and the diameter. More generally, as already alluded to, Liu Hui will qualify as *lü* any set of values sharing this property. Afterwards, he goes on:

(10) If one hence supposes that the surface (*mi*) of the circle fills 3/4 of the surface (*mi*) of the square, the circular cylinder thus also fills 3/4 of the cube.

If, furthermore, one supposes that, the cylinder being represented by the *lü* of the square, 12, what represents the *lü* of the sphere is 9, then, in addition to this, the sphere fills 3/4 of the circular cylinder.

(A computation on fractions to multiply 3/4 by 3/4 follows…)

Therefore the sphere fills 9/16 of the cube. This is why, when one multiplies its volume by 16 and divides by 9, one obtains the volume of the cube.

The diameter of the sphere and the side of the cube are equal, hence, if one divides this by extraction of the cube root, one obtains the diameter.

But this meaning/reasoning (*yi*) is wrong. How to prove (*yan*)[39] this?" (My emphasis)

The most important fact here is that, in a first part of his commentary following this problem, Liu Hui develops a reasoning that accounts for the procedure given by the Canon to solve it, and concludes that this "*yi*" is wrong, a fact that he sets out to prove. Let us sum up the idea of the reasoning just quoted, before drawing some conclusions. The reasoning runs as follows: the procedure amounts, Liu Hui shows, to giving the sphere as 9/16 of the circumscribed cube. If the ratio of the circumference to the diameter is 3 to 1, the inscribed cylinder fills 3/4 of the cube. Then, if the sphere is taken as filling 3/4 of the cylinder, the coefficient of 9/16, expressing the relationship of the sphere to the cube, is obtained. Based on this, the procedure for extracting the spherical root comes from inverting this ratio: multiplying the volume of the sphere by 16 and dividing by 9 yields the volume of the circumscribed cube, whose side equals the diameter of the sphere. Extracting the cube root yields the sought-after value.

This is the *yi* that is said to be wrong, and, in order to criticize it, Liu Hui points out that the solid of whose volume the sphere occupies 3/4 is not the cylinder, but the intersection of two cylinders, both inscribed in the cube, with perpendicular axes. However, he laments being unable to push the reasoning forward and establish the relationship between the volumes of this solid and the cube.[40]

This passage is interesting for us in several respects. In the first place, Liu Hui seems to rely on the context of *The Nine Chapters* as well as on the form of the procedure to formulate the reasoning that yielded the procedure. In fact, his argument exploits the structure of the coefficient 9/16 put into play by the procedure and proceeds from interpreting geometrically that it is the square of 3/4. The essential point for me here is that Liu Hui refers to this whole reasoning that the authors of the procedure may have used to yield the procedure and to account for its correctness with the term "*yi*". This constitutes, in my view, a main clue regarding how he conceives of the nature of the *yi*, this "meaning" or

[39] In Liu Hui's commentary and later mathematical texts, *yan* refers to a kind of proof characterized by its use of visual auxiliaries, "figure" (*tu*) or blocks (*qi*), as is the case here (Chemla, "Qu'est-ce qu'un problème", pp. 120–1).

[40] Wagner, "Liu Hui and Tsu Keng-chih," translates the whole passage, including the proof attributed to Zu Geng that determines this ratio.

"intention," whose contour we attempt to grasp in this context. The *yi* of the Canon, or of the procedures—the understanding of which has elicited the composition of Liu Hui's commentary, he claims in (1)— could hence have taken the form of a reasoning that yielded the procedure recorded in the Canon. As a consequence, this would imply that Liu Hui assumes there are proofs "meant" in the statement of the procedures of the Canon, and his commentary makes them explicit.[41] This fits with his conception of a commentary as "exploring the meaning (*yi*) of the ancients" (passage 2). Li Chunfeng's commentary attests to the same use of the term *yi,* to refer to a reasoning. After having quoted, in his comments on the same problem, the alternative procedure elaborated by Zu Gengzhi for extracting the "spherical root," Li Chunfeng introduces the proof that establishes the correctness of the procedure with the question: "What is its meaning [*yi*]?".

However, to go back to Liu Hui, another conclusion can be drawn from the spherical root passage quoted above. In this case, after having brought to light a "*yi*," Liu Hui discards it by proving its mathematical inadequacy. It is difficult for me to determine whether he discards what he conceives of as being here "the *yi*" attributed to the Canon, or whether he rejects this first interpretation of the Canon. I would be tempted to opt for the first explanation, which would have important consequences as regards our understanding of the practice of exegesis in this case.[42] Whatever the case, it is interesting that we see mathematical reasoning intervening in two ways for the exegesis. First, interpreting the Canon requires writing down a would-be proof. Second, accepting it as *yi* also supposes that reasoning be put into play.

This example illustrates quite adequately some general points concerning proof as carried out in the context of the commentaries to *The Nine Chapters*. Let us stress them, since this will prove useful in what follows.

The deduction quoted above and formulated to account for the "procedure for extracting the spherical root" combines a geometrical reasoning and the writing of the procedure that yields the sought-after result. Let us observe how they correspond to each other. A first geometrical argument shows how, under an assumption corresponding to

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[41] Chemla, "Theoretical Aspects," gives arguments in favor of this hypothesis. As we argue in greater detail below, this interpretation of the nature of the *yi* in the context of mathematics accords with most of its occurrences in the commentary. See my glossary in K. Chemla & Guo S., *Les Neuf chapitres*.

[42] See Chemla, "Theoretical Aspects".

taking π as 3, with a rule of three, one obtains the volume of the cylinder on the basis of the volume of the cube, by multiplying by 3/4. A second geometrical argument reveals the assumption that allows a second rule of three to be applied to establish that the volume of the sphere is obtained as 3/4 of that of the circumscribed cylinder. These two steps constitute the decomposition of the "procedure for extracting the spherical root" into building blocks, whose intention can be made clear in terms of the geometrical situation. The third step of the reasoning consists of reworking the sequence of building blocks established in the first steps, so as to fuse them into a more compact, but equivalent, procedure: multiplying 3/4 by 3/4 provides the fraction 9/16 that allows one to go directly, under the same assumptions, from the circumscribed cube to the sphere. This corresponds to rewriting the first sequence of operations, that is, the succession of two rules of three, whose correctness had been established into a sequence that is closer to the procedure given in the Canon. This rewriting takes a sequence of operations and transforms them into another sequence of operations using what are known to be valid operations. The meaning of the two numbers 9 and 16 is thereby brought to light geometrically. At this point, Liu Hui has shown how to go from the cube to the sphere. The procedure of the Canon requires the inverse transformation. By applying another valid operation—inverting—to the obtained procedure, Liu Hui obtains the procedure that relies on the volume of the sphere to yield that of the circumscribed cube. A last geometrical argument shows how, by appending a cube root extraction to the previous list of operations, one obtains the desired result, the diameter of the sphere. Bringing together the sequence of operations that have been progressively elaborated shows that Liu Hui has attained the procedure of the Canon whose correctness was to be proved, and that he has established that it actually computes the expected result. The correctness of the procedure given by *The Nine Chapters* is thereby established, under the assumptions made explicit during the reasoning.

To recapitulate, the reasoning involves shaping a procedure, the meanings or intentions of which can be made explicit geometrically, step by step. This requires articulating building blocks in a valid way. It also consists of again using various valid operations to gradually transform the procedure that, under an assumption revealed in the second geometrical step, had been shown to yield the correct result into the procedure whose correctness was to be established.[43] The proof hence makes use of algorithms at various levels. This explains why, as we shall see, there are tight relations and exchanges between proof and algorithm. It is the whole process that shapes the procedure of the Canon, which Liu Hui refers to as *yi*, before bringing to light why the assumptions made do not hold true.

Another argument supports this interpretation of *yi*. As we saw, the process involves interpreting the results of some operations geometrically. In proving algorithms that evaluate geometrical magnitudes, Liu Hui regularly makes use of figures (*tu)* for plane geometry or blocks (*qi*) for spatial geometry. As we saw in (1), he introduced the *tu* as a major tool for his commentary: "(...) and if the bodies are dissected (*jie*) with figures (*tu*) (...)". When, while proving the correctness of the algorithm of cube root extraction—a passage that comes immediately before the "procedure for extracting the spherical root"—Liu Hui first introduces the second type of visual auxiliary, the blocks, he justifies it by both quoting the "Great Commentary" (*Xici dazhuan*) on the *Book of Change* (*Yijing*) and referring to his preface: " 'Speech cannot exhaust the "meaning"' (*yi)* (*yan bu jin yi*),[44] hence to dissect/analyze (*jie*) this [volume], one must use blocks, this is the only way to get to understanding [the procedure]." In fact, this statement concludes his proof of the correctness of the algorithm for cube root extraction, in which he has used solid blocks to decompose the cube and interpret the results or the "intentions' of the successive prescribed operations. Among the many links that this statement reveals in Liu Hui's thought, let us stress two.

Blocks were just used by Liu Hui to formulate a proof establishing the correctness of a procedure. The commentator thus connects the introduction of the blocks, in addition to words, with making the "meaning (*yi*)" explicit. Moreover, the production of the proof, with words and blocks, is given as aiming at "understanding" this part of the Canon. Both points confirm the link we established between looking for the *yi* and producing proofs.

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[43] This component of the proofs can be interpreted a kind of "algebraic proof within an algorithmic context," see Karine Chemla, "Fractions and irrationals between algorithm and proof in ancient China", *Studies in History of Medicine and Science*, 1997/98, 15, n° 1–2, New Series: 31–54 and "Algorithmes et histoire de la démonstration," in Régis Morelon et Ahmad Hasnaoui (eds), *De Zénon d'Elée à Poincaré. Recueil d'études en hommage à Roshdi Rashed* (Louvain-La-Neuve: Peteers, 2004), pp. 175–204.

[44] *Xici*, *shang,* chap. 12. In the introduction of chapter 4 and the footnotes to the translation in Chemla & Guo, *Les Neuf chapitres*, the reader can find more detail about this proof of the correctness of the algorithm for extracting cube roots.

These elements of interpretation of the term *yi*, the "meaning" or the "intention" that the commentators seek to explore, are confirmed by another interesting occurrence of the word *yi* in Liu Hui's commentary.[45] To explain this point with some detail, we need to develop a brief analysis of the problem quoted above, in passage (7). It is solved by a procedure (7b), whose correctness the commentator establishes. Let us quote it again:

> (7B) PROCEDURE: ONE PLACES (ON THE COUNTING SURFACE) THE 100 *BU* OF THE GOOD WALKER, AND SUBTRACTS FROM IT THE 60 *BU* OF THE BAD WALKER; **THERE REMAINS 40 *BU***, WHICH IS TAKEN AS DIVISOR. ONE MULTIPLIES, BY THE **100 *BU* OF THE GOOD WALKER**, **THE 100 *BU* THAT THE BAD WALKER HAD FIRST WALKED**, WHICH MAKES THE DIVIDEND. DIVIDING THE DIVIDEND BY THE DIVISOR GIVES THE RESULT IN *BU*. (MY EMPHASIS)

Liu Hui's account of the correctness of the procedure proceeds by bringing to light how, in fact, it amounts to a rule of three. To this end, Liu Hui interprets each of the values used in general terms that form the scheme of the rule of three, which *The Nine Chapters* calls the "procedure of Suppose."[46] In the Canon, the description of the rule of three brings into play three quantities, to which specific and abstract names are attached: the "*lü* of what one has" and the "*lü* of what one seeks" designate the known values that govern the exchange between the quantity of the thing that is possessed—the "quantity of what one has"—and the quantity of the thing into which it is transformed, i.e., the unknown. As already alluded to, qualifying some of these data as *lü* indicates that they possess the property that they can be simultaneously transformed in the same way.

Let us sketch how, by simply naming the values involved, Liu Hui accounts for the procedure (7b). The commentator designates the quantity of 40 *bu*, obtained as the difference between the distances simultaneously described by the good and the bad walkers, as the "*lü* of what the bad walker first walked." In relation to this, the distance of 100 *bu*

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[45] See Chemla, "Algorithmes et histoire de la démonstration," for a detailed analysis of the passage alluded to here. The text is to be found in Qian Baocong, *Suanjing shishu*, vol. 1, pp. 191–2, Chemla & Guo, *Les Neuf chapitres*, pp. 516–9.

[46] See the opening section of Chapter 2 (Qian Baocong, *Suanjing shishu*, vol. 1, p. 114, Chemla & Guo, *Les Neuf chapitres*, pp. 222-5).

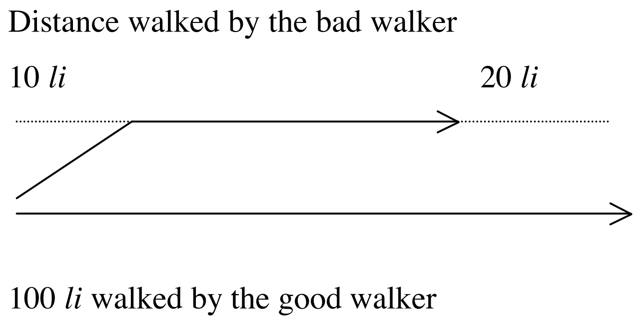
described by the good walker is called the "*lü* of the pursuit and the catch-up." In fact, the third value in the procedure (7b) is the distance first walked by the bad walker. It hence corresponds to the "quantity of what one has," in contrast to the unknown, which is the distance along which the good walker pursues and, in the end, catches up to the bad one. Distributing these names amounts to formulating a proportion, which captures the reasoning that accounts for the correctness of the procedure. It corresponds to stating that the desired distance is described by the good walker during the same time as that necessary to describe the 100 *bu* first walked by the bad walker at a speed corresponding to the difference between the speeds of the two walkers. The essential point around which this proof revolves is the interpretation, in terms of the situation considered by the problem, of the difference in relation to the distance first walked.

In other words, Liu Hui formulates a reasoning that explains what the algorithm carries out, and he *simultaneously* highlights that, in fact, the algorithm formally amounts to putting into play a rule of three. The articulation between interpreting the computations with respect to the situation considered and bringing to light a more general algorithm underlying the examined procedure is an essential characteristic of the proofs as carried out by the commentator. We shall find the same combination in the problem that immediately follows this one in the Canon, in relation to a quite telling occurrence of the term *yi*, whose nature we strive to elucidate.

The following problem in *The Nine Chapters* is different, although the situation described is of the same type. Moreover, accordingly, the algorithm given to solve it differs from the previous one. Let us sketch the problem and algorithm, concentrating on only the elements useful for our discussion here.[47]

> (7C) SUPPOSE THAT A BAD WALKER FIRST WALKS 10 *LI* AND THAT A GOOD WALKER, PURSUING HIM FOR 100 *LI*, HIS ADVANCE ON THE BAD WALKER THEN REACHES 20 *LI*. ONE ASKS IN HOW MANY *LI* THE GOOD WALKER HAD REACHED THE BAD ONE.

[47] Chemla, "Algorithmes et histoire de la démonstration," provides greater detail on this problem 13 of chapter 6 (Qian Baocong, *Suanjing shishu*, vol. 1, p. 192, Chemla & Guo, *Les Neuf chapitres*, pp. 518-20).



> (7d) Procedure: one places (on the counting surface) the 10 *li* that the bad walker had first walked and one **increases** this by the advance of 20 *li* taken by the good walker, which is taken as divisor. One multiplies, by the **10 *li* that the bad walker had first walked, the 100 *li* of the good walker**, which makes the dividend. Dividing the dividend by the divisor gives the result in *li*. (My emphasis)

In this case too, Liu Hui accounts for the correctness of the procedure by highlighting how it brings into play a rule of three. Moreover, he does so in the same way, i.e., by adequately naming, and thereby interpreting step by step, the values in the procedure of *The Nine Chapters*. However, if we compare how he distributes the names in both cases (7b & 7c), an interesting phenomenon appears. In fact, Liu Hui gives the values in the latter procedure the *same names* as those in the former problem. As a consequence, the value obtained by summing the 10 *li* and the 20 *li* is again called the "*lü* of what was first walked." In relation to this, the value of 10 *li* that enters in the computation of the "dividend" appears to be the "quantity of what one has." Moreover, Liu Hui calls the 100 *li* described by the good walker the "*lü* of the pursuit and the catch-up." As previously, the names point to a reasoning that accounts for the correctness of the algorithm. The name chosen for the sum of 10 *li* and 20 *li* brings to light that, like the 100 *bu* in the former problem, the value represents the difference between the lengths of two paths described by the good and the bad walker over the same period of time. The procedure hence appears to state that the sought-after distance is described by the good walker over the same time period as that necessary to describe the 10 *li* first walked by the bad walker at a speed corresponding to the difference between the speeds of both walkers.

In conclusion, Liu Hui again expounds a reasoning showing why this procedure works and highlighting that it too has the structure of a rule of three. However, what is most striking is that the two reasonings exhibited bring to light the fact that, despite superficial differences, both procedures share essential characteristics. They make use of similar features in both situations in the same way. In other words, Liu Hui comments on both procedures from an angle that was chosen to disclose that the same reasoning applies to both and that they both constitute an extension of the same general procedure: the rule of three.

The essential point, now, for us is that the commentator concludes the development relating to the latter procedure by stating: "Its *yi* is like the one for the previous procedure." From this assertion, several points can be made. First, it is again clear that, in Liu Hui's view, the two procedures have a *yi,* "a meaning," which he has made explicit. Moreover, this *yi* again takes the shape of a reasoning that accounts for the correctness of the procedure. This is what can be deduced from what is being compared and declared to be similar. Last but not least, the procedures appear to be different. It is only at the level of their "meaning/*yi*" that their likeness is exhibited and stated.

To sum up, in all these instances, Liu Hui seems to assume that the *yi*(s) of *The Nine Chapters*—the understanding of which, his preface explains, was what prompted the writing of the commentary—can take the form of the reasonings that produced the various algorithms. As a consequence, making reasonings explicit constitutes one of the mathematical practices relating to inquiring into the "meaning/intention (*yi*)" of the Canon. This agrees with what can be found in Liu Hui's and Li Chunfeng's commentaries: after virtually every procedure of the Canon, they formulate proofs of their correctness. Incidentally, this also agrees with the more general use of the term *yi* throughout the commentaries. *Yi* refers to the "meaning," the "intention" of an operation or a subprocedure, in the sense of an interpretation of that which is being computed, formulated in terms of the situation. The overall reasoning that accounts for a procedure requires elucidating and articulating the "meaning" of its subprocedures. Making these *yi*s explicit proves to be an essential aspect in the proof of algorithms. The global *yi* for a procedure hence derives from a combination of these more local ones.[48]

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[48] This is illustrated by the reasoning accounting for the "procedure for extracting the spherical root," as described above. See "Technical terms and expressions linked to proof in ancient Chinese mathematical

<u>Inquiring into the *yi* and exhibiting fundamental procedures</u>

The previous two examples allow us to go one step further towards anwering the questions with which we started by revealing a crucial fact: writing down proofs, or, in other words, <u>exhibiting the *yi*, leads to establishing connections between procedures that, at first sight, seem unrelated</u>. Indeed, the fact that, on the basis of the proofs, unseen connections between procedures can be unveiled appears to be an important motivation for inquiring into the *yi*.

The relation exhibited between the two problems above establishes a link between them at two levels. First, at the level of the situations that are to solved by the procedures, the proofs of their correctness bring to light that the reasonings make use of intermediaries that have the same meaning and they use them in the same way. From a semantic point of view, the procedures thus appear to rely on the same strategy.

Secondly, at a more formal level, the proofs simultaneously highlight that both procedures are instantiations, "extensions," of a more general algorithm—the rule of three. In other words, the procedures appear to share the same strategy at the level of the procedure's form as well.[49] Moreover, this connects them with the variety of procedures that the commentators show also formally amount to putting into play rules of three. Such a situation evokes the relationship of the procedures in Chapter 9 to the "procedure of the base and the height," with which the chapter begins.

This is the very point where we can go back to the questions addressed in this article. In fact, a link is established here between what is brought to light by the proof and relates to the *yi*, on one hand, and Li Chunfeng's declaration (3) on the other hand. Indeed, the rule of three that the proofs reveal is at play in various procedures happens to be one of the four fundamental operations that the seventh-century commentator lists. What is striking here is that if we now turn to the other operations that appear in the same list and in Liu Hui's list (5), i.e., multiplication, division, homogenization and equalization, we realize that these operations present themselves in proofs too, and for some of them, such as homogenization,

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commentaries," Abstract for the workshop "Histoire et historiographie de la démonstration mathématique", 17–19 May 2002, www.columbia.edu/cu/reidhall. The complete paper is in preparation.

[49] On this concept, see Chemla & Guo, *Les Neuf chapitres*, p. 30 or Karine Chemla, "What is at stake in mathematical proofs from third century China?," *Science in Context*, 1997, <u>10</u>:227–51.

only in proofs. Furthermore, in the proofs, they play exactly the same part as we indicated for the rule of three.

Let us outline the case of homogenization and equalization to sketch this point.[50] While proving the correctness of different procedures, Liu Hui regularly interprets the "meaning" or the "intention" (*yi*) of their main operations as "equalizing" (*tong*) some quantities and, in correlation with this, "making" other quantities "homogeneous" (*qi*). The main example is to be found in his commentary on the procedure for adding up fractions (*hefenshu*), in the very context of which, as we saw, Liu Hui made his declaration (5). The algorithm described by the Canon to add up fractions (say, for example, 2/3 and 5/7) prescribes multiplying the denominators by each other, which, in the example, yields 21. It further prescribes multiplying each numerator by the denominator that does not correspond to it: in the example, 2 becomes 14 and 5 becomes 15. The result is thus given as 14+15 to be divided by 21. Liu Hui suggests calling the first operation equalizing, and the second ones homogenizing. In relation to the designations chosen, he brings to light that "multiplying the denominators by each other" equalizes the denominators of all the fractions involved. In addition, he shows that "multiplying each numerator by the denominator that does not correspond to it" makes each numerator homogeneous with the new denominator, which is why one can now add them. The commentator had indicated previously why such operations on fractions were valid. While highlighting why and how the procedure yields the sought-after result, Liu Hui shows that it brings into play an "equalization" and "homogenizations." This would remain unnoticed, were it not that, in (5), Liu Hui designates them as "key points in mathematics" when they first occur and, in relation to this, demonstrates that the same operations are at play when he accounts for the correctness of other procedures. For instance, in Chapter 8, the Canon describes an algorithm to solve systems of *n* simultaneous linear equations with *n* unknowns. Let us sketch it briefly and hence in somewhat less detail than may be necessary for full accuracy.[51] If the system to be solved is

$ax + by = c$

$a'x + b'y = c'$

[50] I cannot fully develop this case here; I shall only allude to it. For a more detailed treatment, see: Chemla, "Theoretical Aspects," "What is at stake," and "Les problèmes comme champ d'interprétation."

[51] Compare Chemla, "Les problèmes comme champ d'interprétation," or Chemla and Guo, *Les Neuf chapitres* for a more accurate account.

the procedure given in *The Nine Chapters* amounts to transforming the equations into

$$aa'x + ba'y = ca'$$
$$aa'x + ab'y = ac'$$

so as to obtain *y* with the equation:

$$(ab'-a'b)y = ac'-a'c$$

Liu Hui recognizes here that the procedure proceeds by equalizing the coefficients of *x* and by homogenizing the other terms of the equations. Again, he has previously accounted for the validity of these operations on equations. By introducing these two designations, the commentator operates at two levels. On the one hand, he indicates the reason why the algorithm works, i.e., why, in fact, there is elimination. However, on the other hand, he discloses a formal strategy at play in the algorithm: equalizing some quantities and homogenizing others. This reveals a connection between the procedures for solving systems of equations and for adding fractions. Equalizing and homogenizing are exhibited in several other contexts. However, these two examples suffice to illustrate what is at stake in all cases.

The interpretation in terms of "equalization" and "homogenization" allows Liu Hui to establish that these different procedures are correct. The quantities that are made equal differ in each of these contexts, and the reasons why, once the values are equalized, the procedure can be shown to be correct, differ. However, at the formal level, as in the previous example with the rule of three, the proofs reveal an operational pattern common to all of them. Proofs bring to light that, to yield the desired result, the procedures follow the same formal strategy in the way they rely on the situation in which they operate. This is expressed by the recurrence of the pair of operations "equalize and homogenize." As a consequence, in the same way as above, through proofs, Liu Hui unveils links between procedures of *The Nine Chapters* that, at first sight, seemed unrelated. Procedures given by the Canon and apparently unconnected are revealed, through the proof, as being mere instantiations, or "extensions," of the same fundamental procedures. These essential procedures can be understood as the "origin" or the "essential points"[52] of the procedures given by the Canon.

In a similar way, we observe that multiplication and division, often by virtue of their relation of opposition, are regularly exhibited to be at play in the way in which a procedure correctly operates. They enter the list of formal strategies available to yield a procedure.[53] In my view, this is what primarily justifies that they be compared to the rule of three or to the procedure of "equalization and homogenization."

It is highly interesting that the commentators specifically reserved another term, *yi'*, to designate, for a procedure, the kind of "meaning" that comes from bringing to light the fundamental operation underlying it. One hence regularly reads that a procedure has "the *yi'* of homogenization and equalization" or the "*yi'* of the rule of three."[54]

To sum up, our commentators' practice of proof can be characterized as follows: through accounting for the correctness of the procedures, the proofs bring to light that the various procedures ultimately put into play a limited number of formal strategies or, in other terms, fundamental operations. These fundamental operations can all present themselves in the different steps we distinguished in the proof—in the building blocks that enter in the making of a procedure, as well as in the way they are combined and rewritten.

In this way, the proofs reduce what *The Nine Chapters* gave as a diversity of procedures. And, a crucial connection emerges between the search for the *yi*, as carried out through formulating proofs, and the elements entering the architectures of mathematical knowledge described by our commentators: the limited number of formal strategies that are brought to light by the proofs are the very operations selected to enter into Liu Hui's and Li Chunfeng's lists of fundamental procedures. Note that these operations can be identified

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[52] A remark is in order here. The fact that "the procedure of the base and the height" is not included in the list of fundamental operations seems to indicate that, even though it constitutes the source of the procedures of Chapter 9, its level of generality cannot compare to the procedures that the commentators include in their declarations. We shall come back to this point in a future publication.

[53] Chemla, "Fractions and irrationals," illustrates this point.

[54] Such statements can be found in Liu Hui's as well as in Li Chunfeng's commentaries. See my glossary, in K. Chemla & Guo S., *Les Neuf chapitres*. It is interesting to note that this interpretation of *yi'* agrees with Fung Yu-lan's interpretation of Wang Bi's use of the term, in relation to *li* and *yi* (Fung Yu-lan, *A History of Chinese Philosophy*. Translated by Derk Bodde, 2 vols. (Princeton: Princeton University Press, 1952–3), vol.2, pp. 185–7, quoted on the basis of the 1973 reprint).

through writing down the proofs not separately, but as a whole, and through continuously confronting them as we did above. We provided evidence that clearly demonstrates that Liu Hui compared procedures on the basis of their *yi*.

Also note the convergence between what comes out here through confronting the proofs and the meaning the commentator read, as we saw above, in the organization of Chapter 9: the identification of general operations whose efficiency extends as widely as possible appears to be a concern that permeates his reading of the Canon throughout.

<u>Interpreting the commentators' fundamental declarations</u>

At this point, we are in a position to offer an interpretation of the declarations of our commentators and to elucidate the expectation that *The Nine Chapters* may be all encompassing. Moreover, we can also suggest how this expectation relates to describing an architecture of mathematical knowledge.

Let us start with Li Chunfeng's statement (3). The previous interpretation of the link between the nature of the commentators' proofs and the emergence of fundamental operations fits with the structure and the wording of his declaration (3). All the procedures of *The Nine Chapters*, he claims, make use of four fundamental operations—this, in my view, is what writing down the proofs, or exhibiting the *yi* of the Canon as he and his predecessor did, brings to light. Li Chunfeng lists the general procedures that he considers to be fundamental based on the proofs he and the previous commentators made explicit for the whole Canon.

Li Chunfeng extends this: he *deduces* that these fundamental procedures exhaust the <u>methods</u> of all mathematical procedures. Notice that, although his target clearly encompasses all mathematical algorithms, his claim does not bear on the procedures themselves, but on their "methods." I suggest that, by this term, Li Chunfeng refers to the overall procedure that, as I explained above in the case of the extraction of the spherical root, constitutes the reasoning producing, i.e., accounting for, the algorithm. The "method" would include the steps of exhibiting the building blocks to be combined to reach the result, as well as of articulating and rewriting them to yield the algorithm that carries out the envisioned task. We shall see below other pieces of evidence supporting this interpretation.

Li's claim could hence be understood as follows: the proofs carried out by the commentators within the framework of *The Nine Chapters* bring to light all the fundamental procedures that could appear in the proof, or in the shaping, of any mathematical algorithm. This leads me to an interpretation of the expectation that the Canon is complete. *The Nine Chapters* would be all encompassing in the sense that the procedures it contains point to all the fundamental operations needed in the production of mathematical algorithms. Moreover, the algorithms point to the fundamental operations, in the sense that proving their correctness, while bringing the proofs into confrontation with each other, brings the general operations to light. In other words, borrowed from Liu Hui's preface (1), *The Nine Chapters* allows "synthesizing the source of mathematical procedures."

This interpretation would explain why the statement of comprehensiveness is always articulated upon the description of an architecture of mathematics. The commentator, by means of proofs and comparing proofs establishes a list of the fundamental operations indirectly indicated by the Canon and common to several of its algorithms. The operations constitute the basis of the structure of mathematical knowledge, which explains why these architectures all display the same features.

In addition to this, Li Chunfeng distinguishes himself by placing at the root of the structure a kind of object, the *lü*, which he identifies as that which explains the efficiency of the fundamental operations.[55]

The Canon would indicate these fundamental operations by displaying some of the diverse and paradigmatic manifestations they can take. Moreover, the Canon would indicate how the basic patterns present themselves and combine in a diversity of situations—a syntagmatic dimension, if we will—showing how their efficiency can be extended to virtually any situation. For example, adding up fractions and solving systems of simultaneous linear equations, when compared, reveal the fundamental pattern of equalizing and homogenizing. However, a detailed comparison would show that the algorithms differ in the ways they use the fundamental patterns.

The task of the commentator would then be to understand how these manifestations reveal the basic patterns involved in the shaping of mathematical procedures, the "source" from which all mathematics flow. It is interesting that the fundamental operations that constitute the core of the commentators' most fundamental declarations about *The Nine Chapters* are considered as expressing the *yi'* of procedures, a word traditionally associated

[55] Compare Chemla, "Algorithmes et histoire de la démonstration".

with the "significance" of a canon. In addition to this, it is highly revealing, in this respect, that the commentators diverge in their lists of fundamental operations. This shows that there is not a unique way of conceiving of this list. However, this also discloses that, despite divergences, commentators carry out the same task through their exegesis.

If we understand that bringing to light how the fundamental operations are at play in a given procedure amounts to exhibiting its *li*, its "inner constitution," its basic structure, then we can interpret Liu Hui's preface (1) and Yang Hui's statement (4) in similar ways.[56]

In the case of Yang Hui, this interpretation is straightforward. If the "stem" from which, according to Liu Hui's declaration (1), mathematical procedures diverge is understood as consisting of the list of fundamental operations (5), then we can interpret in the same way that he moves from "analyzing the *li*" of the procedure to considering *The Nine Chapters* as all-encompassing. In Liu Hui's view, the "meaning/*yi*" of the procedures would reveal that their *li* consists of articulating only the same limited number of fundamental operations and combining them according to the same principles. Therefore, stating that the Canon is complete would amount to believing that any mathematical procedure could be produced with the same building blocks and syntagmatic principles.

This line of interpretation receives confirmation from a statement included in the postface to *The Nine Chapters* by the 1213 editor of the Tang collection *Ten Canons of Mathematics*, Bao Huanzhi. He states:

> (11) Among the books of mathematical procedures, there are altogether ten schools. One can only consider *The Nine Chapters* as being the head of the Canons. With the methods of its nine parts of mathematics (*jiu shu*), there is nothing which is not complete. Although the procedures established by the various schools present

[56] It is quite interesting that the description of the task required of candidates for State examinations in mathematics, according to the *Xin Tang shu*, made use of such terms as *li* and *yi'* (for an interpretation, see Siu Mankeung & Volkov, Alexei, "Official curriculum in traditional Chinese mathematics: How did candidates pass the examinations," *Historia scientiarum*, 1999, 9-1:85–99, on pp. 92 ff.). The requirement could be understood in a more precise way, if we agree on interpreting these terms as I suggest. This would support Alexei Volkov's thesis that the composition and selection of commentaries may have related to specific modes of mathematical education.

variation, when one looks for the original meaning (*yi*), they all come from them. (My emphasis)

If we follow Bao Huanzhi, *The Nine Chapters* is not all encompassing in the sense that the Canon would contain all mathematical procedures. Bao explicitly refers to procedures that may not be included in it. However, when it comes to their original meaning—an expression that combines the idea of source and that of *yi*, two ideas whose connection has already been noted—then the original meaning should necessarily fall within the scope of *The Nine Chapters*. One may understand that, when inquiring into the reasoning that leads to establishing, or yielding the procedure, the procedure thereby would either appear as an instantiation of a general fundamental operation or amount to a combination of fundamental operations that can derived in a way from the Canon. The proofs of the procedures then would reveal how, in fact, they fall within the scope of the Canon.

CONCLUSION

Let us summarize the main points made in this article. I have argued that the commentators' explicit attempt to explore the *yi*, the "meaning/intention" of the Canon, led them to systematically formulate proofs of the correctness of the procedures it contained. However, the practice of mathematical proof put into play in relation to exegesis was quite specific. Such proofs seem to have been the tool for inquiring at a formal level into connections between apparently unrelated procedures of *The Nine Chapters*. These connections appear to have brought to light fundamental procedures shared by the algorithms of the Canon. This was the main means the commentators used to identify fundamental algorithms from which all the procedures of the Canon derive and whose efficiency would extend the farthest. Valuing the search for such algorithms can be best understood in a context where generality is granted much weight.[57]

Such a practice of exegesis reveals the commentators' conception of how the Canon expresses its meaning. I tried to show that the commentators agreed in the following respect:

[57] See Chemla, "Generality above abstraction."

in their view, *The Nine Chapters* displays a variety of procedures that, although simple, are rich enough, when confronting their proofs, to indicate the "origin," the "essential points," and how they can extend to virtually any situation. The Canon reveals its source through a variety of procedures that derive from the source.

This may explain how exegesis led the commentators to describe a structure of the world of mathematical procedures in which all procedures are understood to flow from a limited number of fundamental and very general operations. Although this modality of architecture shares the outer shape of a tree with an axiomatico-deductive structure, it is clear that the principles through which the trees are shaped differ fundamentally.

This may also help us understand the ways in which the Canon could be expected to be all encompassing. *The Nine Chapters* was believed to highlight the fundamental operations for producing any procedure. In Bao Huanzhi's terms, even if a procedure appeared to exceed the scope of the Canon, looking at its "original meaning" would reveal how, in fact, by extension or combination of fundamental operations, it still fell within the range of *The Nine Chapters*.

The specific practice of proof carried out by the commentators may hence appear to be the essential element linking the statement of the completeness of the Canon and the conceptions of the architecture of mathematical reality.

Appendix 1:

What is our earliest evidence regarding the status of *The Nine Chapters* as a "Canon (*jing*)"?

Present-day historiography of mathematics in China generally holds the view that it was only in 656, and not before, that *The Nine Chapters* acquired the status of a "*jing*." According to this view, this change in status occurred when *The Nine Chapters* was included in the *Ten Canons of Mathematics*, a compilation with commentary by Li Chunfeng and his assistants; the project had been imperially commissioned and, in 656, was presented to the throne. It was within this context that the commentary by Li Chunfeng on *The Nine Chapters* was composed, as part of the commentary that he and his assistants wrote on the set of ten Canons. Immediately after the production of this collection, the Canons were used as textbooks for teaching mathematics in the College of Mathematics within the State

University.[58] It is hence regularly assumed that the Canons were compiled to be used as textbooks to be used in the University.

To be sure, there is clear evidence showing that these books were used for teaching. It is also beyond doubt that within the collection of the ten Canons, the title of *The Nine Chapters on Mathematical Procedures* was modified into *The* Canon *of the Nine Chapters of Mathematics* (*Jiuzhang suan* *jing*). In the same way, the title of the oldest book included in the collection, *The Gnomon of the Zhou* (*Zhoubi*), whose composition historians date to the first century either BCE or CE, was also changed into *The Mathematical* Canon *of the Gnomon of the Zhou* (*Zhoubi suanjing*). However, there are reasons to distinguish the fact that the books were considered "Canons (*jing*)" from the fact that their title was changed to stress this fact. Although the title was changed for the 656 edition, my first claim is that there is evidence showing that these two oldest texts included in the compilation were designated as *jing* long before this date.[59] My second claim is that we must also distinguish the fact that the Canons were used in the University from the idea that making a set of Canons was motivated only by the need for textbooks for the University (point 2).

Since there is considerable confusion on this question, I shall attempt to clarify the matter, starting with my second point. As is shown in Siu & Volkov, "Official curriculum in traditional Chinese mathematics," the ten Canons were not the only textbooks used in the university to teach mathematics. In addition, two other books were used that *did not belong to the collection*. Let us leave aside the one that is no longer extant, and concentrate on the second one, the *Memoir on the Procedures of Numbering* (*Shushu jiyi*). Its composition is attributed to Xu Yue around 220 CE, and Zhen Luan (fl. ca. 560) wrote a commentary on it.[60] However, according to the extant evidence, there is *no commentary by Li Chunfeng* on

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[58] See Siu & Volkov, "Official curriculum in traditional Chinese mathematics."

[59] More generally, it is clear that the title of the most important of all Canons, that is, the Confucian Canons, did not always include the term *jing*.

[60] On the attribution and the content, see Volkov Alexei, "Large numbers and counting rods," in Alexei Volkov (ed.), *Sous les nombres, le monde. Extrême-Orient, Extrême-Occident,* 1994, 16: 71–92. In the thirteenth century, the book had already been lost, when Bao Huanzhi, the thirteenth-century editor of the *Ten Canons of Mathematics*, found a copy in a Taoist temple. As he explains in his postface (Song edition, pp. 10–1), in relation to the fact that the book was used in the Tang University together with the Canons, he made the decision to include it in his reprint of *The Ten Canons*. Incidentally, Bao Huanzhi's postface is extremely clear on the fact that in his view, the ten Canons are books that are of a nature different from Xu Yue's *Memoir*.

it. The facts that, on the one hand, it was not part of the collection and that, on the other hand, no commentary on it by Li Chunfeng exists reinforce each other and imply that at the very time when the collection of *The Ten Canons of Mathematics* was compiled, there was a distinction between belonging to the set of Canons and being a textbook in the University. In other words, we cannot equate being a Canon with being a textbook. Something more is indicated by granting a book the status of a Canon than simply designating it as teaching material.

Let me now turn to my first point and discuss the evidence I found showing that the two oldest Canons, *The Gnomon of the Zhou* and *The Nine Chapters*, were perceived as *jing* earlier than 656. I consider the earliest occurrence of the term *jing* in relation to *The Nine Chapters* to be a sentence in Liu Hui's preface, which reads: "the procedures of the Canon got scattered and damaged 經術散壞 *jing shu san huai*" (see excerpt 1). This claim can be contested, because the common understanding of the expression *jingshu* is the study of the Confucian classics. However, within the context of mathematics, *shu* designates a "procedure," as is the case in the title: *The Nine Chapters on Mathematical Procedures*. Such procedures form the core of this mathematical writing. Moreover, as can be seen in quotation (1), in the context in which the sentence occurs, the previous and the following sentences both relate to mathematics. Further, the problem discussed soon afterwards in the preface is that raised by the restoring of *The Nine Chapters* from pieces of old text. Lastly, from a semantic point of view, I think that the combination of verbs "scatter" and "damage," which follows the expression, is better suited to something material than to the study of the Classics. As a result, I find it more plausible that in the commentator's preface the expression refers to "the procedures of the Canon." However, if that were my only piece of evidence, the argument would possibly not be strong enough. To this I can add evidence of two kinds.

First, the preface of another one of *The Ten Canons*, the *Mathematical Canon Continuing the Ancients* (*Qigu suanjing*), refers twice to *The Nine Chapters* with the term

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Following in the same tradition, Qian Baocong, *Suanjing shishu*, vol. 2, pp. 535–48, includes an edition of the *Memoir on the Procedures of Numbering*.

jing.[61] The author, Wang Xiaotong, refers to Liu Hui, whose preface he quotes. Like his predecessor, he considers that the scripture, which he compares to the *Book of Change (Yijing*), has suffered from damage and that its gaps must be filled up. This provides us with evidence that *The Nine Chapters* was considered a *jing* before 656.

At the same time, it is important that we can find two clear and early references to the other book from the beginning of the Common Era, *The Gnomon of the Zhou*, that designate it as *jing*. The first occurrence is to be found in the preface from the third-century commentator Zhao Shuang.[62] The second one occurs in the second layer of commentary by Zhen Luan (fl. ca. 560), who refers to what he is commenting on with the term *jing*.[63]

As mentioned in the introduction, this essay would like to suggest that we ought to investigate what this designation meant, without assuming that the answer will be the same for all books and all readers. It falls outside of our scope to fulfill the entire program. Let me stress that such a program cannot always be developed: for some of the mathematical Canons, we do not have substantial enough evidence of how earlier readers approached them to inquire into the question. Yet, despite such limitations, I urge that we refrain from anachronistic approaches to these texts. Moreover, it is clear that in the case of *The Gnomon of the Zhou*, the evidence of its status as a Canon is abundant, and I hope to be able to devote a publication to it in the future.

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[61] See the critical edition in Qian Baocong, *Suanjing shishu*, vol. 2, pp. 493–527. The text of the preface is on pp. 493–4. In his introduction to the text (p. 487), Qian argues that, even though the book was composed much earlier, the preface must have been completed some time after 626.

[62] We refer to the critical edition of *The Gnomon of the Zhou* in Qian, *Suanjing shishu*, vol. 1, pp. 11–80. In the preface by Zhao Shuang, one reads: "I relied on the Canon (*jing*) to make figures." (Qian, *Suanjing shishu*, vol. 1, p. 11).

[63] Qian Baocong, *Suanjing shishu*, vol. 1, p. 39. See the entry "*jing* Canon" in my glossary, Chemla & Guo, *Les Neuf chapitres*, p. 942.