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Untying the Knot:
How Einstein Found His Way Back to Field Equations Discarded in the Zurich Notebook

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“She bent down to tie the laces
of my shoes.
Tangled up in blue.”
—Bob Dylan

1. INTRODUCTION: NEW ANSWERS TO OLD QUESTIONS

Sometimes the most obvious questions are the most fruitful ones. The Zurich Notebook is a case in point. The notebook shows that Einstein already considered the field equations of general relativity about three years before he published them in November 1915. In the spring of 1913 he settled on different equations, known as the “Entwurf” field equations after the title of the paper in which they were first published (Einstein and Grossmann 1913). By Einstein’s own lights, this move compromised one of the fundamental principles of his theory, the extension of the principle of relativity from uniform to arbitrary motion. Einstein had sought to implement this principle by constructing field equations out of generally-covariant expressions.[1] The “Entwurf” field equations are not generally covariant. When Einstein published the equations, only their covariance under general linear transformations was assured. This raises two obvious questions. Why did Einstein reject equations of much broader covariance in 1912-1913? And why did he return to them in November 1915?

A new answer to the first question has emerged from the analysis of the Zurich Notebook presented in this volume. This calls for a reassessment of Einstein’s subsequent elaboration of the “Entwurf” theory and of the transition to the theory of November 1915. On the basis of a reexamination of Einstein’s papers and correspondence of this period, we propose a new answer to the second question.

For the discussion of these matters, it is important to distinguish between two strategies for finding suitable gravitational field equations, a ‘physical strategy’ and a

1 Throughout the period covered by this paper, Einstein labored under the misconception that general covariance automatically extends the principle of relativity from uniform to arbitrary motion. He did not appreciate the difference between the role of Lorentz invariance in special relativity and the role of general covariance in general relativity (see Norton 1999 for an illuminating discussion of this conflation). For discussion of Einstein’s subsequent attempt to implement general relativity through what he called “Mach’s principle,” see the editorial note, “The Einstein–De Sitter–Weyl–Klein Debate,” in CPAE 8, 351–357.

'mathematical strategy'. Following the physical strategy, one constructs field equations in analogy with Maxwell's equations, making sure from the start that they satisfy energy-momentum conservation and that they reduce to the Poisson equation of Newtonian theory in the case of weak static fields. This is the approach that originally led Einstein to the "Entwurf" field equations. Following the mathematical strategy, one picks candidate field equations based largely on considerations of mathematical elegance and then investigates whether they make sense from a physical point of view.[2] With hindsight, one easily recognizes that the latter approach provides a royal road to the generally-covariant field equations of November 1915. Einstein himself used a combination of the two strategies. In the Zurich Notebook, he tried the mathematical strategy first, ran into what appeared to be insurmountable difficulties, switched to the physical strategy, and ended up with the "Entwurf" field equations. On that much all scholars working in this area agree. The question is what Einstein did in late 1915. The currently standard answer is that he abandoned the physical strategy, went back to the mathematical strategy prematurely abandoned in the Zurich Notebook, and in short order produced the happy results of November 1915.[3] With very few exceptions, Einstein's pronouncements—both at the time and in retrospect years later—fit very well with this answer.

As the title of our paper suggests, however, we see no abrupt change of strategy in 1915. Our metaphor is not "cutting the knot" but "untying the knot." We argue that Einstein found the field equations of general relativity by changing one element in a formalism he had developed in 1914 encoding the various physical considerations that had gone into the derivation of the "Entwurf" field equations. He picked a new mathematical object, known as the Christoffel symbols, to represent the gravitational field. This one modification, it turned out, untangled the knot of conditions and definitions that his theory had become in 1914–1915. We thus argue that the field equations of general relativity were the fruit of Einstein's relentless pursuit of the physical strategy. That is not to say that the mathematical strategy did not play any role at all. Without it Einstein would not have recognized that his new definition of the gravitational field was the key to the solution of his problem of finding suitable field equations. What happened in 1915 was that the physical strategy led Einstein back to field equations to which the mathematical strategy had already led him in the Zurich Notebook but which he had then been forced to reject since he could not find a satisfactory physical interpretation for them. With two routes to the same field equations, Einstein had the luxury of a choice in how to present them to the Berlin Academy. He went with the mathematical considerations, which he turned into a simple and effective argument for his new field equations. It would have been much more complicated and less persuasive to opt for an exposition faithful to the arduous journey he himself had

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2 For more careful discussion of the distinction between the 'mathematical strategy' and the 'physical strategy', see secs. 1.1 and 5.1 of "Commentary …" (this volume).

3 See, e.g., (Norton 1984, 142), (Janssen 1999, 151), (Van Dongen 2002, 30). The most explicit version of this account is given in (Norton 2000). We shall have occasion to quote some typical passages from this paper in sec. 10.

been forced to undertake only to discover in the end that equations he had considered very early on were the right ones after all.[4] In the context of discovery, the physical argument had been primary and the role of the mathematical argument had been to reinforce that argument. In the context of justification, it was just the other way around. Einstein gave pride of place to the mathematical argument and used elements from his physical argument only to show that his new field equations were perfectly acceptable on physical grounds. Once the mathematical route to the field equations had been reified in his first communication to the Berlin Academy of November 1915, the physical route rapidly faded from memory. The streamlined argument of the context of justification quickly supplanted the messy reasoning of the context of discovery. Einstein succumbed to a typical case of selective amnesia. Before long he had eyes only for the mathematical strand in his reasoning and had lost sight of the physical strand altogether.[5]

In the remainder of this introduction, we give an outline, as non-technical as possible, of our new understanding of the path that took Einstein away from generally-covariant field equations and back again in the period 1912–1915. The emphasis will be on the second part of this fascinating tale. The case for our new reconstruction is strong but largely circumstantial. We shall highlight the most important pieces of evidence in the introduction, so that the reader can judge for him- or herself how well our reconstruction is supported by the documents without having to go through the detailed calculations that make up the balance of the paper.

1.1 Tying the Knot: Coordinate Restrictions

The central problem frustrating Einstein's search for generally-covariant field equations in the Zurich Notebook was his peculiar use of what a modern relativist would immediately recognize as coordinate conditions. One needs to impose such conditions, which the metric tensor has to satisfy in addition to the field equations, if one wants to compare equations of general relativity, which are valid in arbitrary coordinates, to equations in Newtonian gravitational theory, which in their standard form are valid only in inertial frames. One needs a coordinate condition, for instance, to show that the generally-covariant field equations of general relativity reduce to the Poisson equation of Newtonian theory in the case of weak static fields (see, e.g., Wald 1984, 75–77). Such conditions are not essential to the theory. One can pick whatever coordinate condition is most convenient for the problem at hand. From a modern point of view this is trivial and there would be no point in spelling it out, if it were not for the fact that Einstein's use of such additional conditions both in the Zurich Notebook and in his subsequent elaboration of the "Entwurf" theory deviated sharply

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4 A paper written by one of us (JR) which we have cannibalized for this paper was therefore called "Progress in a Loop."

5 Jeroen van Dongen (2002, 46–47) has emphasized that Einstein's selective memory only served him all too well in his later years in his defense of relying on a purely mathematical strategy in the search for a unified field theory.

from our modern use.

As Einstein was examining various generally-covariant expressions in 1912-1913 to determine whether physically acceptable field equations could be extracted from them, he assumed that he needed additional conditions not just to recover the Poisson equation for weak static fields but also to guarantee that the equations be compatible with the law of energy-momentum conservation.[6] In general relativity energy-momentum conservation is a direct consequence of the general covariance of the Einstein field equations (see Einstein 1916c and sec. 9). This result is an instantiation of one of Emmy Noether's celebrated theorems connecting symmetries and conservation laws (Noether 1918).[7] In 1912–1913, however, Einstein thought that energy-momentum conservation required that the covariance of the field equations be restricted. In the Zurich Notebook he did not make a clear distinction between conditions imposed to guarantee energy-momentum conservation and conditions imposed to recover the Poisson equation for weak static fields. On the contrary, once he had found a condition that accomplished the latter, he would investigate what further conditions, if any, were needed for the former.[8]

Einstein used these conditions to eliminate various terms from equations of broad covariance and looked upon the truncated equations of severely restricted covariance rather than upon the equations of broad covariance he started from as candidates for the fundamental field equations of his theory. Since coordinate conditions used in this manner are ubiquitous in the Zurich Notebook we introduced a special name for them. We call them *coordinate restrictions*.[9]

This notion is the key to understanding why Einstein did not publish field equations based on the Riemann tensor in 1913. He recognized that the Riemann tensor—or rather the Ricci tensor, a direct descendant of it—was the natural starting point for finding field equations, but he had great difficulty finding coordinate restrictions that

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6 This is a lesson Einstein had learned the hard way earlier in 1912 when he had been forced to modify the field equations of his theory for static gravitational fields because the original equations violated energy-momentum conservation (Einstein 1912, sec. 4). For further discussion, see "Pathways …" (this volume).

7 For careful discussion of Noether's theorems and some simple but informative applications of them, see (Brading 2002); for a discussion of how they emerged from the discussion of general relativity in Göttingen, see (Renn and Stachel 1999), (Rowe 1999), and (Sauer 1999). For a concise discussion of Einstein's ideas about energy-momentum conservation in the period 1912–1918, see sec. VIII of the introduction to CPAE 8.

8 This approach is clearly in evidence on pp. 19L–20L of the Zurich Notebook.

9 See secs. 4.1 of "Commentary …" (this volume). There is no agreement among the authors of this volume as to why Einstein used coordinate restrictions. The majority view is that Einstein at the time did not yet have the modern understanding of coordinate conditions. John Norton, however, argues that Einstein did have the modern understanding all along and offers a different explanation for why he nonetheless chose to use coordinate restrictions instead (see "What was Einstein's fatal prejudice?," this volume). The story we tell in this paper is compatible with both views. This is an indication of how difficult it is to decide between them. John Norton argues that it boils down to one's view of Einstein's *modus operandi*. For the record, we share the view presented in "What did Einstein know …" (this volume).

would guarantee compatibility with energy-momentum conservation even in first approximation. Moreover, none of the coordinate restrictions with which he could recover the Poisson equation for weak static fields left him enough covariance to implement the equivalence principle and the generalized principle of relativity.[10]

So towards the end of the notes on gravity in the Zurich Notebook, Einstein switched from the mathematical to the physical strategy. Instead of starting from a mathematical object such as the Ricci tensor with well-defined covariance properties, he now started from the physical requirements that the field equations reduce to the Poisson equation for weak static fields and that they be compatible with energy-momentum conservation. Instead of demanding broad covariance, he only demanded covariance under general linear transformations.[11] From these requirements he derived the equations that would serve as the fundamental field equations of his theory without bothering to find the generally-covariant equations of which these equations would be the truncated version or the coordinate restriction with which to do the truncating. Einstein convinced himself that this procedure led to a unique result: the "Entwurf" field equations.

1.2 Tightening the Noose: Covariance Properties of the "Entwurf" Field Equations

Einstein's further elaboration of the "Entwurf" theory in 1913–1914 centered on clarifying the covariance properties of the "Entwurf" field equations. In August 1913, he produced an argument purporting to show that because of energy-momentum conservation the equations' covariance group had to be limited to general linear transformations. This argument was published in an addendum to the journal version of the "Entwurf" paper (Einstein and Grossmann 1914a). Within a few months, Einstein realized that it was based on a faulty premise.[12] The idea that energy-momentum conservation circumscribes the covariance of the field equations nonetheless survived. In

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10 John Norton (1984, 102, 111–112, 142-143) argued that the incompatibility of the harmonic coordinate condition with the spatially flat metric that Einstein thought should describe weak static fields plays a crucial role both in Einstein's rejection of field equations based on the Ricci tensor in 1912–1913 and in his choice of new field equations in the first paper of November 1915 (Einstein 1915a). We seriously doubt whether Einstein was even aware of this incompatibility either at the time of the Zurich Notebook or in November 1915. We see no evidence that this incompatibility played any role in Einstein's search for gravitational field equations (cf. sec. 5.4 of "Commentary …" [this volume] and the conclusion of sec. 5 below). Einstein's prejudice about the form of the metric for weak static field, for which there is abundant textual evidence, did play a role—as Norton (1984, 146–148) also emphasized—in that it was incompatible with field equations containing a term with the trace of the energy-momentum tensor of matter. Aside from general covariance, this trace term is the most important feature distinguishing the Einstein field equations from their "Entwurf" counterpart (see sec. 7 and the appendix).

11 Einstein's hope was that the equations would also be invariant under what he later called "non-autonomous" transformations to accelerating frames of reference. See sec. 3.3 below for discussion of the concept of non-autonomous transformations.

12 For discussion of this episode, see (Norton 1984, sec. 5), "Pathways …" (this volume), and sec. 2 of "What did Einstein know …" (this volume).

the same paper in which he retracted his fallacious argument (Einstein and Grossmann 1914b), he and Grossmann presented a new argument tying the covariance of the "Entwurf" field equations to energy-momentum conservation. With Noether's theorems still four years into the future, Einstein's intuition that energy-momentum conservation is closely related to the covariance of the field equations is quite remarkable. It will play a crucial role in our story.

Einstein and Grossmann (1914b) found four conditions, compactly written as $B_\mu = 0$, that in conjunction with the "Entwurf" field equations imply energy-momentum conservation. They then used a variational formalism to show that these same conditions determine the covariance properties of the "Entwurf" field equations. Einstein thought that these conditions were the coordinate restriction with which the "Entwurf" field equations could be extracted from generally-covariant equations.[13] He had no interest in finding the latter, since his infamous 'hole argument' had meanwhile convinced him that the field equations could not possibly be generally covariant.[14] In fact, Einstein and Grossmann claimed that the four conditions they had found gave the field equations the maximum covariance allowed by the hole argument. With these results, the theory appeared to have reached its definitive form.

In the spring of 1914, Einstein left Zurich and Grossmann and moved to Berlin. In October 1914, nearly three months into the Great War, he completed a lengthy review article on his new theory, no longer called "a general*ized* theory of relativity" (Einstein and Grossmann 1913) but "the *general* theory of relativity" (Einstein 1914c). In this article, he reiterated and tried to improve on the results of his second paper with Grossmann. He now used the variational formalism to deal both with the covariance properties of the field equations and with energy-momentum conservation. And he did so without specifying the Lagrangian ahead of time as he had done in the paper with Grossmann. He only assumed that the Lagrangian transforms as a scalar under general linear transformations. He found that a generic version of the set of conditions he had found with Grossmann, still written as $B_\mu = 0$, is necessary both for the covariance of the field equations and for their compatibility with energy-momentum conservation. Energy-momentum conservation, however, called for an additional set of conditions, compactly written as $S_\sigma^\nu = 0$. Einstein believed that these extra conditions uniquely picked out the Lagrangian giving the "Entwurf" field equations. As a matter of fact they do no such thing.

1.3 At the End of His Rope: The Demise of the "Entwurf" Field Equations

In early 1915, the Italian mathematician Tullio Levi-Civita contested some of the results of Einstein's review article but Einstein did not give ground.[15] Curiously, the

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13 Einstein (1914b, 178) believed that there is a corresponding generally-covariant equation for any physically meaningful equation that is not. See note 57 below for the relevant passage.

14 For a discussion of (the origin of) the hole argument, see sec. 4 of "What did Einstein know …" (this volume).

correspondence with Levi-Civita did not touch on Einstein's uniqueness argument, even though Levi-Civita's interest in the article had been triggered by a letter from Max Abraham complaining about the arbitrariness of Einstein's choice of the Lagrangian (Cattani and De Maria 1989, 185). It was not until October 1915, that Einstein himself realized that his uniqueness argument was illusory. This setback came hard on the heels of another one. He had discovered that the "Entwurf" field equations are incompatible with one of the guiding ideas of the theory—the idea that the inertial forces of rotation can be conceived of as gravitational forces. Michele Besso had already put his finger on this problem two years earlier, but Einstein had ignored his friend's warnings.[16] He finally faced up to the problem in September 1915.

In a letter to H. A. Lorentz of October 12, 1915 (CPAE 8, Doc. 129), Einstein explained where his uniqueness argument went wrong. The extra conditions $S_\sigma^\nu = 0$ that he had used to determine the Lagrangian are trivially satisfied by any Lagrangian invariant under general linear transformations. So both sets of conditions—$B_\mu = 0$ and $S_\sigma^\nu = 0$—needed for energy-momentum conservation also emerge from the analysis of the theory's covariance properties. From a modern point of view, this is just an instance of one of Noether's theorems. If one sets the Lagrangian in Einstein's variational formalism equal to the Riemann curvature scalar, as Einstein (1916c) himself would do the following year, the four conditions $B_\mu = 0$ become the contracted Bianchi identities.

1.4 Pulling a Thread: from the "Entwurf" Field Equations to the November Tensor and the Einstein Field Equations

Despite the problem of rotation and the evaporation of the uniqueness argument, Einstein was not ready to part with the "Entwurf" field equations just yet. He told Lorentz that they are still the only equations that reduce to the Poisson equation for weak static fields. Just a few weeks later, however, on November 4, 1915, he submitted a short paper to the Berlin Academy in which he replaced the "Entwurf" field equations by equations based on the Riemann tensor. He had examined and rejected these exact same equations three years earlier in the Zurich Notebook. Three more short communications to the academy followed in rapid succession, two of them with further modifications of the field equations (Einstein 1915b, d) and one on the perihelion motion of Mercury (Einstein 1915c). By the end of November, Einstein had thus arrived at the generally-covariant field equations that still bear his name and he had solved an outstanding puzzle in planetary astronomy.

What happened those last few weeks of October? Einstein has left us some tantalizing clues. In the first November paper, he singled out one element and called it "a fateful prejudice."[17] In a letter written later that month, shortly after the dust had set-

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15 See the correspondence between Einstein and Levi-Civita in March–May 1915 in CPAE 8.

16 For discussion of Einstein's struggles with the problem of rotation in 1913–1915, see sec. 3 of "What did Einstein know …" (this volume).

tled, he wrote that changing that one element had been "the key to [the] solution."[18] The element in question is the definition of the components of the gravitational field. In the "Entwurf" theory, they are essentially just the derivatives of components of the metric field.[19] This is the straightforward generalization of the definition of the gravitational field in Newtonian theory as the gradient of the gravitational potential. In Einstein's theory the components of the metric field play the role of the gravitational potentials. In the final version of the theory, the gravitational field is represented by the so-called Christoffel symbols. The Christoffel symbols consist of a sum of three terms with derivatives of the metric.[20] These objects play an important role in Riemannian geometry. They also occur in the geodesic equation, which makes them the natural candidates for representing the gravitational field. Surprisingly from a modern point of view, the first November paper is the first place where Einstein actually makes this observation. Why did he put so much emphasis all of a sudden on the definition of the gravitational field?[21]

In the "Entwurf" theory, both the field equations and the equation for energy-momentum conservation were originally formulated in terms of the metric, the quantity representing the gravitational potentials, not in terms of the quantity representing the gravitational field.[22] Einstein, however, also tried to write both equations in terms of the field. In this form, the analogy between the "Entwurf" theory and electrodynamics, which Einstein had consciously pursued in constructing the theory,[23] was brought out more clearly. This in turn made the physical meaning of the equations more perspicuous.

In August 1913, Einstein had found that the "Entwurf" field equations can be written in the form of Maxwell's equations, with the four-divergence of the field on the left-hand side and the field's sources—the sum of the energy-momentum densities of matter and gravitational field[24]—on the right-hand side (Einstein 1913, 1258, eq. 7b). He had written the equations in this form ever since.[25] In the Zurich notebook he had already noticed that the term representing the gravitational force density

17 "ein verhängnisvolles Vorurteil" (Einstein 1915a, 782). Cf. note 38 below.

18 "Den Schlüssel zu dieser Lösung …" Einstein to Arnold Sommerfeld, 28 November 1915 (CPAE 8, Doc. 153). Cf. note 39 below.

19 More precisely: derivatives of the covariant metric contracted with the contravariant metric.

20 More precisely: derivatives of the covariant metric contracted with the contravariant metric.

21 Without the analysis of the Zurich Notebook presented in this volume, Einstein's remarks about the definition of the gravitational field have, as John Norton (1984, 145) put it, "all the flavor of an after-the-fact rationalization." Norton was also right in that these comments do not help us understand why Einstein turned his back on equations extracted from the Riemann tensor in 1913.

22 As John Stachel (2004) points out in his "Newstein" paper, this is in part because Einstein had to make do with the mathematics available to him. Far from providing all the tools he needed, differential geometry at the time still lacked the concept of an affine connection, which is a much more natural object than the metric to describe the inertio-gravitational field of general relativity. The absence of the notion of parallel displacement and the concept of an affine connection also tripped up H. A. Lorentz in 1916 when he tried to give a coordinate-free formulation of general relativity (Janssen 1992).

23 For further discussion of the role of this analogy, see "Pathways …" (this volume).

in the energy-momentum balance equation can be interpreted as an inner product of the field and its sources, just like the Lorentz force that some extended charge distribution experiences from its self-field. Most importantly, in his review article of 1914, Einstein wrote the Lagrangian for the "Entwurf" field equations in terms of the components of the gravitational field (Einstein 1914c, 1076, note 1). The "Entwurf" Lagrangian is the same quadratic expression in the field as the Lagrangian for the free Maxwell field. These structural similarities to electrodynamics—in the field equations and in the expressions for the force density and the Lagrangian—carry over from the "Entwurf" theory to the theory of the November 1915 papers if the components of the gravitational field are redefined as the Christoffel symbols.[26]

In view of this continuity, Einstein's remark that the new definition of the gravitational field was "the key to the solution" suggests a natural pathway along which, sometime during the second half of October 1915, Einstein found his way back to field equations of broad covariance discarded three years earlier in the Zurich Notebook.

Not long after his letter to Lorentz of October 12, Einstein must have come to accept that the problem of rotation was the nemesis of the "Entwurf" field equations (Janssen 1999). He needed new field equations or rather a new Lagrangian from which such new equations could be derived. His variational formalism would give him the conditions to guarantee compatibility with energy-momentum conservation. The analogy with electrodynamics could be used to narrow the range of plausible candidates for the new Lagrangian. Replacing the derivatives of the metric field by the Christoffel symbols as components of the gravitational field in the "Entwurf" Lagrangian may well have been one of the first things he tried.

The expression for the left-hand side of the field equations that one finds upon feeding this new Lagrangian into the variational formalism bears a striking resemblance to the left-hand side of field equations that Einstein had extracted from the Ricci tensor in the Zurich Notebook by imposing the (relatively weak) restriction to unimodular transformations. These are transformations with a Jacobian equal to one, or, equivalently, transformations under which the determinant g of the metric trans-

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24 Because of the equivalence of energy and mass (inertial and gravitational) it is clear that the gravitational field contributes to its own source. This, of course, is a major disanalogy between the gravitational field equations and Maxwell's equations. For one thing, unlike Maxwell's equations, the gravitational field equations will not be linear in the components of the field.

25 See, e.g., (Einstein 1914a, 289, eq. 5), (Einstein 1914b, 179, eq. 6), (Einstein and Grossmann 1914b, 217, eq. II), and (Einstein 1914c, 1077, eq. 81). For the original form of the "Entwurf" field equations, see (Einstein and Grossmann 1913, 15–17, eqs. 13–16, 18, and 21).

26 Both in the "Entwurf" theory and in the theory of the first November paper, the Lagrangian has the form $g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$, where $\Gamma^{\beta}_{\alpha\nu}$ are the components of the gravitational field. In the "Entwurf" theory, $\Gamma^{\beta}_{\alpha\nu} \equiv \frac{1}{2}g^{\beta\tau}g_{\tau\alpha,\nu}$; in the first November paper, $\Gamma^{\beta}_{\alpha\nu} \equiv -\left\{ \begin{matrix} \beta \\ \alpha\nu \end{matrix} \right\} \equiv -\frac{1}{2}g^{\beta\tau}(g_{\tau\alpha,\nu} + g_{\tau\nu,\alpha} - g_{\alpha\nu,\tau})$. The Lagrangian is modelled on the Lagrangian for the free Maxwell field, $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$, where $F_{\mu\nu}$ and $F^{\mu\nu}$ are the covariant and contravariant components of the electromagnetic field, respectively.

forms as a scalar. Imposing the restriction to unimodular transformations on the general variational formalism, which allows one to omit a factor of $\sqrt{-g}$ in the action, and feeding the new Lagrangian into this version of the formalism, one finds that the left-hand side of the resulting field equations is exactly the same as the left-hand side of field equations discarded in the Zurich Notebook. Because of its reappearance in November 1915, we call this expression the *November tensor.*[27]

By adjusting the physical reasoning that had gone into the derivation of the "Entwurf" field equations, Einstein had thus found new field equations that could also be derived along the lines of the mathematical strategy. This was exactly the sort of convergence of physical and mathematical considerations that had eluded Einstein in the Zurich Notebook and in his work on the "Entwurf" theory. The best he had been able to do was to convince himself in 1914 that the "Entwurf" field equations can at least in principle be extracted from generally-covariant ones with the help of the coordinate restriction $B_\mu = 0$. Now physical and mathematical considerations both pointed to the November tensor. He set the November tensor equal to the energy-momentum tensor for matter, and confidently replaced the "Entwurf" field equations by these new equations in his first communication to the Berlin Academy of November 1915.

In the Zurich Notebook, Einstein had not been able to prove compatibility of field equations based on the November tensor with energy-momentum conservation. His variational formalism, even though it had to be used with caution because of the restriction to unimodular transformations, now provided all the guidance he needed to solve that problem. This unexpected windfall, however, brought a new puzzle. Having caught on to the connection between covariance and conservation laws, Einstein had come to expect that the covariance of the field equations was determined by the four conditions $B_\mu = 0$ in his variational formalism that at the same time guarantee energy-momentum conservation. The covariance of the November tensor, however, is much broader than these conditions would seem to allow. What did Einstein make of this apparent mismatch between covariance and conservation laws? The November 1915 papers again provide some important clues.

In the first of these papers (Einstein 1915a, 785), Einstein rewrote the four conditions $B_\mu = 0$ in such a way that they can be replaced by one stronger condition. He then showed that this stronger condition can be replaced by the requirement that the determinant of the metric not be a constant. In the second and in the fourth paper, Einstein proposed ways to circumvent this requirement. These moves become readily understandable if we assume that they were made in response to the discrepancy between covariance and conservation laws mentioned above. Given that the November tensor is invariant under arbitrary unimodular transformations, Einstein expected that energy-momentum conservation would not require any further restrictions. As we mentioned above, g, the determinant of the metric, transforms as a scalar under unimodular transformations. This explains why Einstein tried to rewrite the standard

27 See sec. 5.5 of "Commentary …" (this volume).

four conditions $B_\mu = 0$ giving energy-momentum conservation as one condition on g. It also explains why he was not satisfied with the requirement that g not be a constant. The restriction to unimodular *transformations* only requires g to transform as a scalar, not that it be either a constant or a variable. In fact, it turns out to be advantageous to impose the stronger restriction to unimodular *coordinates*, i.e., coordinates in which $g = -1$. It is thus perfectly understandable that Einstein tried to replace the condition that g not be a constant by the condition that $g = -1$. This was the driving force behind the transition from the field equations of the first November paper to those of the fourth one. In this last paper of November 1915, Einstein showed that one arrives at the desired condition $g = -1$ if a term involving the trace of the energy-momentum tensor is added to the field equations of the first November paper. These equations can be looked upon as generally-covariant equations expressed in terms of unimodular coordinates. The generally-covariant equations are the Einstein field equations.

1.5 Untying the Knot: Coordinate Conditions

Two problems that had defeated the November tensor in the Zurich Notebook still need to be addressed. How did Einstein show that his new field equations reduce to the Poisson equation for weak static fields and that they allow Minkowski space-time in rotating coordinates? These are two separate problems and they are easily solved separately, but in the Zurich Notebook they had become entangled with one another and with the problem of energy-momentum conservation. The entanglement was the result of Einstein's use of coordinate restrictions. One and the same restriction had to reduce the field equations to the Poisson equation for weak static fields, guarantee energy-momentum conservation, and allow the metric for Minkowski space-time in rotating coordinates. Coordinate conditions only have to do the first of these three things. The three problems can thus be disentangled by switching from coordinate restrictions to coordinate conditions.

Einstein, we believe, made this switch when he saw that field equations based on the November tensor can be made compatible with energy-momentum conservation by imposing just one weak coordinate restriction. Recovering the Poisson equation for weak static fields still required the usual four restrictions. This discrepancy of one restriction versus four opened up the possibility to handle recovery of the Poisson equation with a coordinate condition in the modern sense and impose a coordinate restriction only for energy-momentum conservation. It is impossible to say whether Einstein had arrived at the modern understanding of coordinate conditions earlier or whether he only reached this point when faced with this unexpected discrepancy. Only the separation of the two sets of conditions, however, made it possible to put the modern understanding of coordinate conditions to good use. Not only could Einstein now decouple the problem of energy-momentum conservation from the problem of recovering the Poisson equation, he could also decouple the latter from the problem of rotation. It is this disentanglement of various conditions and requirements that we tried to capture in the title of our paper: “Untying the knot.”

The first November paper contains the first unambiguous instance of Einstein applying a coordinate condition in the modern sense to show that the field equations reduce to the Poisson equation for weak static fields (Einstein 1915a, 786). In the Zurich Notebook Einstein had used what we call the *Hertz restriction*[28] for this purpose. One of the problems with this restriction was that it does not allow the Minkowski metric in rotating coordinates. In the first November paper, Einstein used the exact same mathematical formula, but now interpreted as a coordinate condition rather than a coordinate restriction. As Einstein clearly recognized, it then no longer is a problem that the condition is not satisfied by the Minkowski metric in rotating coordinates. Right after he applied the Hertz condition, he pointed out that the class of unimodular transformations under which the field equations are invariant allow transformations to rotating coordinates. The obvious implication is that the new theory steers clear of the problem of rotation that had defeated the old one.

Einstein had untied the knot. The definition of the components of the gravitational field had been the thread he had pulled to do so. No wonder that he called the old definition “a fatal prejudice” and the new one “the key to the solution.”

1.6 Tug of War: Physics or Mathematics?

How well does the text of the November 1915 papers support our reconstruction of how Einstein found his way back to generally-covariant field equations? As a matter of fact, Einstein does *not* introduce the new field equations by pointing out that they can be obtained simply by changing the definition of the gravitational field in the expression for the Lagrangian from which he had earlier derived the “Entwurf” equations. Instead, he uses the fact that the new field equations are closely related to the generally-covariant Riemann tensor, rehearsing the argument that had led him to the November tensor in the Zurich Notebook. At first glance, this looks like a strike against us. On closer examination, it is not such a clear call. In his paper, Einstein was presumably concerned with making the strongest possible case for his new field equations. No matter how Einstein had arrived at these new field equations, it clearly was more convincing to show that these equations can easily be extracted from the Ricci tensor than to show that they can be obtained by a natural adjustment of the formalism that Einstein had used the year before in his failed attempt to prove the uniqueness of the “Entwurf” field equations. Emphasizing the former argument and suppressing the latter would have been the obvious preemptive strike against skeptical readers who might want to remind him of that fiasco. But it need not even have been a calculated rhetorical move on Einstein’s part. He himself probably saw the connection to the Riemann tensor as the most convincing evidence in favor of his new

28 The only reason for this name is that the condition is discussed in Einstein to Paul Hertz, August 22, 1915 (CPAE 8, Doc. 111). See sec. 5.5.2 of “Commentary …” (this volume).

field equations. It thus makes perfect sense that this is what he emphasized in his presentation and that he only availed himself of the variational formalism to do the one thing he did not know how to do any other way, namely proving compatibility with energy-momentum conservation.

If we are right, Einstein's papers of November 1915 not only gave his contemporaries and a host of later commentators a misleading picture of how he found the field equations of general relativity, they also and most importantly fooled their own author. Einstein would soon forget that he had arrived at the new field equations pursuing the physical strategy and that the complementary mathematical strategy had served mainly to give him the confidence that he was finally on the right track. In his later years, Einstein extolled the virtues of a purely mathematical approach to theory construction. As John Norton (2000) and, in much greater detail, Jeroen van Dongen (2002, 2004) have shown, the older Einstein routinely claimed that this was the lesson he had drawn from the way in which he had found general relativity. The way Einstein remembered it, physics had led him astray; it was only after he had decided to throw in his fate with mathematics that he had found the right theory. In our reconstruction, however, Einstein found his way back to generally-covariant field equations by making one important adjustment to the "Entwurf" theory, a theory born almost entirely out of physical considerations. He saw that he could redefine the components of the gravitational field without losing any of the structural similarities to electrodynamics that made the "Entwurf" theory so attractive from a physical point of view. After a few more twists and turns, this path led him to the Einstein field equations. That mathematical considerations pointed in the same direction undoubtedly inspired confidence that this was the right direction, but guiding him along this path were physical not mathematical considerations.

1.7 The Red Thread: Einstein's Variational Formalism

In the rest of this paper, we fill in the details of our new reconstruction of the transition from the "Entwurf" theory to general relativity. For those who do not want to go through the derivations, we give short summaries at the beginning of all (sub-)sections of the results derived in them. In sec. 2, we review the one result we need from the Zurich Notebook, namely the extraction of the November tensor from the Ricci tensor. In sec. 3, we give a self-contained exposition of the variational formalism of (Einstein 1914c) that plays a pivotal role in our account. In sec. 4, we show how Einstein used this formalism to make what he considered his most compelling case for the "Entwurf" theory. In secs. 5–7, we analyze how Einstein used the formalism in his papers of November 1915 (Einstein 1915a, b, d). In secs. 8 and 9, we turn to two papers (Einstein 1916a, 1916c) in which the results of November 1915 were consolidated, again with the help of the formalism of 1914. In sec. 10, we address the discrepancy noted above between how Einstein presented and remembered his discovery of general relativity and how he actually discovered it. Finally, in the appendix, drawing on calculations scattered throughout the body of the paper, we present a concise and sanitized version of the transition from the "Entwurf" field equations to the Einstein field equations, which makes the relation between these two sets of equations more perspicuous.

2. THE NOVEMBER TENSOR IN THE ZURICH NOTEBOOK

We review how the field equations based on the November tensor in (Einstein 1915a) made their first appearance in the Zurich Notebook. Einstein extracted the November tensor from the Ricci tensor by imposing a restriction to unimodular transformations. He then showed how the Hertz restriction reduces the November tensor to the d'Alembertian acting on the metric in the case of weak fields.

On p. 22R of the Zurich Notebook—at the instigation, it seems, of his friend and collaborator Marcel Grossmann whose name appears at the top of the page—Einstein wrote down the Ricci tensor in the form

$$T_{il} = \frac{\partial}{\partial x^l}\begin{Bmatrix} k \\ ik \end{Bmatrix} - \frac{\partial}{\partial x^k}\begin{Bmatrix} k \\ il \end{Bmatrix} + \begin{Bmatrix} \lambda \\ ik \end{Bmatrix}\begin{Bmatrix} k \\ \lambda l \end{Bmatrix} - \begin{Bmatrix} \lambda \\ il \end{Bmatrix}\begin{Bmatrix} k \\ \lambda k \end{Bmatrix}, \tag{1}$$

where

$$\begin{Bmatrix} k \\ il \end{Bmatrix} \equiv \frac{1}{2}g^{k\alpha}(g_{i\alpha,l} + g_{l\alpha,i} - g_{il,\alpha}) \tag{2}$$

are the Christoffel symbols.[29] Einstein extracted an expression from the Ricci tensor that transforms as a tensor under unimodular transformations. Under such transformations, the quantity

$$T_i \equiv \begin{Bmatrix} k \\ ik \end{Bmatrix} = (\lg\sqrt{-g})_{,i} \tag{3}$$

(with g the determinant of $g_{\mu\nu}$), transforms as a vector, and its covariant derivative,

29 We have adopted a notation that lies somewhere between slavishly following the original text and translating everything into modern language. Our guiding principle has been to use a notation that makes the equations both easy to follow for those familiar with the standard notation of modern general relativity and easy to compare with the original sources for those who want to check our claims against Einstein's own writings. On this basis, we have adopted the following rules. We typically follow Einstein's choice of letters for quantities and indices in the document under discussion. E.g., the Ricci tensor is not written as $R_{\mu\nu}$, as it is in most modern texts, but as T_{il} in our discussion of the Zurich Notebook in this section and as G_{im} in our discussion of (Einstein 1915b) in sec. 7. As in Einstein's writings of this period, all indices, Greek and Latin, run from 1 through 4. However, we do not follow Einstein's idiosyncratic convention before (Einstein 1914c) of writing nearly all indices downstairs and distinguishing between covariant and contravariant components (e.g., the components $g_{\mu\nu}$ and $g^{\mu\nu}$ of the metric) by using a Latin letter for one ($g_{\mu\nu}$) and a Greek letter for the other ($\gamma_{\mu\nu}$). We use Latin letters for all quantities and write all covariant indices downstairs and all contravariant indices upstairs. Following Einstein, we use Fraktur for tensor densities (e.g., $\mathfrak{T}^{\mu}_{\nu} = \sqrt{-g}T^{\mu}_{\nu}$). Deviating from Einstein, we occasionally use commas and semi-colons for ordinary and covariant differentiation, respectively. We consistently use the summation convention (introduced in Einstein 1916a, 788).

$$\frac{\partial T_i}{\partial x^l} - \left\{ \begin{matrix} \lambda \\ il \end{matrix} \right\} T_\lambda = \frac{\partial}{\partial x^l} \left\{ \begin{matrix} k \\ ik \end{matrix} \right\} - \left\{ \begin{matrix} \lambda \\ il \end{matrix} \right\} \left\{ \begin{matrix} k \\ \lambda k \end{matrix} \right\} , \tag{4}$$

transforms as a tensor. It follows that the quantity

$$T^x_{il} \equiv \frac{\partial}{\partial x^k} \left\{ \begin{matrix} k \\ il \end{matrix} \right\} - \left\{ \begin{matrix} \lambda \\ ik \end{matrix} \right\} \left\{ \begin{matrix} k \\ \lambda l \end{matrix} \right\} , \tag{5}$$

which is (minus) the difference between the generally-covariant Ricci tensor in eq. (1) and the expression in eq. (4), also transforms as a tensor under unimodular transformations. This is the quantity we call the *November tensor.*

Setting the November tensor equal to the energy-momentum tensor, $T_{\mu\nu}$, multiplied by the gravitational constant κ, one arrives at the field equations of Einstein's first paper of November 1915 (Einstein 1915a, 783, eq. 16a):

$$\frac{\partial}{\partial x^\alpha} \left\{ \begin{matrix} \alpha \\ \mu\nu \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \mu\beta \end{matrix} \right\} \left\{ \begin{matrix} \beta \\ \alpha\nu \end{matrix} \right\} = \kappa T_{\mu\nu} , \tag{6}$$

The first term on the left-hand side does not reduce to the d'Alembertian acting on the metric in the weak-field case. There are additional terms with unwanted second-order derivatives of the metric. At the time of the Zurich Notebook, this made the November tensor itself unacceptable as a candidate for the left-hand side of the field equations. Einstein, however, extracted a candidate for the left-hand side of the field equations from the November tensor by imposing what we call the *Hertz restriction*,

$$g^{k\alpha}_k \equiv \frac{\partial g^{k\alpha}}{\partial x^k} = 0 . \tag{7}$$

Expanding the Christoffel symbols in the first term of the November tensor, one finds

$$\frac{\partial}{\partial x^k} \left\{ \begin{matrix} k \\ il \end{matrix} \right\} = \frac{1}{2}(g^{k\alpha}(g_{i\alpha,l} + g_{l\alpha,i} - g_{il,\alpha}))_k .$$

Using the Hertz restriction and the relation

$$g^{k\alpha}(g_{i\alpha,l} + g_{l\alpha,i}) = -g^{k\alpha}_l g_{i\alpha} - g^{k\alpha}_i g_{l\alpha}$$

(where $g^{\kappa\alpha}_l \equiv g^{\kappa\alpha}{}_{,l}$), one can rewrite this expression as:

$$\frac{\partial}{\partial x^k} \left\{ \begin{matrix} k \\ il \end{matrix} \right\} = -\frac{1}{2}(g^{k\alpha} g_{il,k\alpha} + g^{k\alpha}_l g_{i\alpha,k} + g^{k\alpha}_i g_{l\alpha,k}) . \tag{8}$$

The first term in parentheses reduces to $-\Box g_{il}$ for weak fields. The other two terms are quadratic in first-order derivatives of the metric like the contribution coming from the term in the November tensor quadratic in the Christoffel symbols. All these terms can be neglected for weak fields.

On p. 23L of the notebook, Einstein tried to bring down the number of terms quadratic in first-order derivatives of the metric in his field equations by introducing yet another coordinate restriction in addition to the Hertz restriction and the restriction to unimodular transformations. We call this new restriction the ϑ-restriction.[30] Einstein discovered that this new coordinate restriction could be used to eliminate the unwanted terms with second-order derivatives as well, so that there was no longer any need for the Hertz restriction. Einstein eventually abandoned the ϑ-restriction because the ϑ-restriction—like the Hertz restriction for that matter—ruled out transformations to rotating frames in Minkowski space-time. After a few more twists and turns, Einstein settled on the "Entwurf" field equations. The November tensor and the Hertz restriction—used now as a coordinate condition—only reappeared in November 1915.

3. EINSTEIN'S VARIATIONAL FORMALISM: FIELD EQUATIONS, ENERGY-MOMENTUM CONSERVATION, AND COVARIANCE PROPERTIES

We cover various aspects of the variational formalism that Einstein used both in his review article on the "Entwurf" theory of October 1914 and in a number of papers on general relativity in 1915–1918. The Lagrangian is left unspecified, so all results hold both in the "Entwurf" theory and in modern general relativity. The main point of the section is to show how two very different lines of reasoning—one aimed at finding conditions to ensure energy-momentum conservation, the other aimed at finding coordinate transformations leaving the action invariant—lead to the exact same conditions on the Lagrangian, written as $S_\sigma^\nu = 0$ *and* $B_\mu = 0$. *The convergence of these two lines of reasoning confirmed what Einstein had come to suspect in the fall of 1913, namely that energy-momentum conservation is directly related to the covariance of the gravitational field equations.*

In early November 1915, Einstein (1915a) replaced the "Entwurf" field equations of severely limited covariance by field equations just a few tweaks away from the generally-covariant Einstein field equations of (Einstein 1915d). The way Einstein initially described it, it was a wholesale replacement:

> After all confidence in *the result and the method* of the earlier theory had thus given way, I saw clearly that a satisfactory solution could only be found in a connection to the general theory of covariants, i.e., to Riemann's covariant.[31]

About six weeks later, he recognized that the old had not been all bad:

> The series of my papers on gravitation is a chain of erroneous paths, which nonetheless gradually brought me closer to my goal.[32]

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30 For discussion of the ϑ-restriction, see note 69 below and secs. 5.5.4–5.5.10 in "Commentary …" (this volume).

It is true that Einstein discarded some of his earlier *results*, but he retained the *method* that he had used to obtain those results. This method is the variational formalism first presented in (Einstein and Grossmann 1914b) and further developed in the definitive exposition of the "Entwurf" theory (Einstein 1914c, part D). In the latter paper, he used this formalism to produce an elegant derivation of the "Entwurf" field equations, to investigate their covariance properties, and to prove their compatibility with energy-momentum conservation. We argue that he used this same formalism to find the successor to the "Entwurf" field equations, published in the first of his four communications to the Prussian Academy in November 1915 (Einstein 1915a).

What complicates the use of the formalism both in the four November papers (Einstein 1915a, b, c, d) and in the first systematic exposition of the new theory (Einstein 1916a) is a restriction to unimodular transformations in the first paper and the choice of unimodular coordinates[33] in the other four. In all these papers, Einstein nonetheless relied heavily on the formalism to guide him in his analysis of the relation between field equations and energy-momentum conservation.

It was only in (Einstein 1916c), written in October 1916, that Einstein first presented the new theory entirely in arbitrary rather than in unimodular coordinates. This paper follows the exposition of the variational formalism in (Einstein 1914c) almost to the letter (as already emphasized in Norton 1984, 141). In early 1918, Einstein used the formalism again to defend his approach to energy-momentum conservation in general relativity against objections from Levi-Civita, Lorentz, Klein, and others (Einstein 1918d). The formalism can also be found in Einstein's lecture notes for a course on general relativity in Berlin in 1919 (CPAE 7, Doc. 19, [pp. 13–17]).

Einstein's reliance on this variational formalism thus provides an important element of continuity in the transition from the "Entwurf" theory to general relativity and puts the lie to Einstein's remark to Sommerfeld that he had lost all confidence in

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31 "Nachdem so jedes Vertrauen im Resultate und Methode der früheren Theorie gewichen war, sah ich klar, dass nur durch einen Anschluss an die allgemeine Kovariantentheorie, d.h. an Riemanns Kovariante, eine befriedigende Lösung gefunden werden konnte." Einstein to Arnold Sommerfeld, November 28, 1915 (CPAE 8, Doc. 153; our emphasis). Unless otherwise noted, all translations are based on those in the companion volumes to the Einstein edition. This letter to Sommerfeld provides the most detailed account of the developments of November 1915 that culminated in the publication of the Einstein field equations and the explanation of the anomalous motion of Mercury's perihelion. This document, however, needs to be treated with care. It was a calculated move on Einstein's part to tell Sommerfeld the whole story rather than, say, Lorentz, with whom he had corresponded much more intensively on matters general relativistic. In the fall of 1915, Sommerfeld was kept apprised of developments not only by Einstein but also by Hilbert. Writing to Sommerfeld, Einstein probably first and foremost wanted to make sure that Sommerfeld knew that he had put his house in order without any help from Hilbert.

32 "Die Serie meiner Gravitationsarbeiten ist eine Kette von Irrwegen, die aber doch allmählich dem Ziele näher führten." Einstein to H. A. Lorentz, January 17, 1916 (CPAE 8, Doc. 183). The mixing of metaphors ("Kette von Irrwegen") is Einstein's, not ours.

33 Recall the discussion of the difference between coordinate restrictions and coordinate conditions in the introduction.

both “the result and the method” of the old theory.

In this section, we cover various aspects of Einstein’s formalism: the derivation of the field equations (sec. 3.1), the treatment of energy-momentum conservation (sec. 3.2), and the investigation of covariance properties (sec. 3.3). In subsequent sections, we discuss the applications of the formalism in the period 1914–1916. In sec. 4, we examine the relevant portion of (Einstein 1914c) published in November 1914. In secs. 5–7, we turn to the papers of November 1915 documenting the transition from the “Entwurf” theory to general relativity (Einstein 1915a, b, d). In sec. 8, we present the streamlined version of the argument of November 1915 given in the review article completed in March 1916 (Einstein 1916a, part C). In sec. 9, we cover what is probably the most elegant application of the formalism, the demonstration that energy-momentum conservation in general relativity is a direct consequence of the general covariance of the field equations. This argument was made in (Einstein 1916c), presented to the Prussian Academy in November 1916. Our story thus covers a timespan of two years, from November 1914 to November 1916. Our main focus will be on the tumultuous developments of one month in the middle of this period, November 1915.

3.1 ***Field equations.*** *With the appropriate definition of the gravitational energy-momentum pseudo-tensor, the field equations can be written in a form resembling Maxwell’s equations, with the divergence of the gravitational field on the left-hand side and the sum of the energy-momentum densities of matter and gravitational field on the right-hand side.*

Consider the gravitational part of the action

$$J = \int Q d\tau , \tag{9}$$

where $Q = H\sqrt{-g}$ is the gravitational part of the Lagrangian,[34] H is some as yet completely undetermined function of $g^{\mu\nu}$ and its first derivatives $g^{\mu\nu}_{\alpha}$, and $d\tau = d^4x$ is a four-dimensional volume element. The condition that J be an extremum, $\delta J = 0$, leads to the Euler-Lagrange equations

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial Q}{\partial g^{\mu\nu}} = 0 . \tag{10}$$

Although his proof did not satisfy Levi-Civita, Einstein thought he could show that the expression on the left-hand side transforms as a tensor density under all transformations under which H transforms as a scalar.

He generalized the vacuum field equations (10) to

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial Q}{\partial g^{\mu\nu}} = -\kappa\sqrt{-g}T_{\mu\nu} \tag{11}$$

34 Strictly speaking, Q is the Lagrangian density (for detailed discussion, see Wald 1984, Appendix E).

in the presence of matter described by the energy-momentum tensor $T_{\mu\nu}$. This equation can be written in a form analogous to Maxwell's equations, with the divergence of the gravitational field on the left-hand side and the sources of the field, the energy-momentum densities of matter and gravitational field, on the right-hand side. Contraction with $g^{\nu\lambda}$ of the left-hand side of eq. (11) gives:

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\right) - g^{\nu\lambda}_{\alpha}\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}} - g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}}\,;$$

contraction with $g^{\nu\lambda}$ of the right-hand $-\kappa\sqrt{-g}T^{\lambda}_{\mu}$. If the gravitational energy-momentum pseudo-tensor t^{λ}_{μ} is defined as

$$\kappa\sqrt{-g}t^{\lambda}_{\mu} \equiv -g^{\nu\lambda}_{\alpha}\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}} - g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}}\,, \tag{12}$$

then eq. (11) can be rewritten as

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\right) = -\kappa(\mathrm{T}^{\lambda}_{\mu} + \mathrm{t}^{\lambda}_{\mu})\,, \tag{13}$$

where $\mathrm{T}^{\lambda}_{\mu} \equiv \sqrt{-g}T^{\lambda}_{\mu}$ and $\mathrm{t}^{\lambda}_{\mu} \equiv \sqrt{-g}t^{\lambda}_{\mu}$ are mixed tensor densities. These field equations fulfill an important requirement: the energy-momentum of the gravitational field enters the source term in the same way as the energy-momentum of matter.

If the quantity in parentheses on the left-hand side of eq. (13) is identified as the gravitational field, the equations have the same structure as Maxwell's equations, $\partial_{\mu}F^{\mu\nu} = \mu_0 j^{\nu}$, where $F^{\mu\nu}$ is the electromagnetic field tensor, μ_0 is a constant, and j^{ν} is the charge-current density, the source of the electromagnetic field.

3.2 ***Energy-momentum conservation.*** *In addition to the field equations, it is assumed that the covariant divergence of the energy-momentum tensor of matter vanishes. This equation can be rewritten as the vanishing of the ordinary divergence of the sum of the energy-momentum tensor densities for matter and gravitational field, provided that the gravitational energy-momentum pseudo-tensor is defined appropriately. Compatibility of this definition and the definition in the preceding subsection leads to the conditions* $S^{\nu}_{\sigma} = 0$ *on the Lagrangian. In 1914, Einstein (erroneously) thought that these conditions uniquely pick out the "Entwurf" Lagrangian. Einstein imposed four more conditions, written as* $B_{\mu} = 0$. *Taken together with the field equations, the conditions* $B_{\mu} = 0$ *imply energy-momentum conservation. In general relativity, these conditions turn into the contracted Bianchi identities.*

The energy-momentum balance for matter in a gravitational field can be written as[35]

$$\mathrm{T}^{\alpha}_{\mu,\alpha} - \begin{Bmatrix} \beta \\ \mu\alpha \end{Bmatrix}\mathrm{T}^{\alpha}_{\beta} = 0\,. \tag{14}$$

This equation is equivalent to (cf. Einstein 1914c, 1056, eq. 42a):[36]

$$\mathrm{T}^{\alpha}_{\mu,\alpha} - \frac{1}{2} g^{\beta\rho} g_{\rho\alpha,\mu} \mathrm{T}^{\alpha}_{\beta} = 0 . \tag{15}$$

The four-momentum density of matter at any given point can only change in two ways: it can flow to or from neighboring points and it can be transferred to or from the gravitational field at that point. The first term in eqs. (14)–(15) describes the former process, the second term the latter. The second term gives the rate at which four-momentum density is transferred from gravitational field to matter. This term thus represents the gravitational force density. The analogy with the Lorentz force density, $f_{\mu} = j^{\nu} F_{\mu\nu}$, suggests that this quantity should be equal to the contraction of the components of the field (i.e., the gravitational analogue of the electromagnetic field tensor $F_{\mu\nu}$) and the components of the field's source (i.e., the gravitational analogue of the charge-current density j^{ν}). So the gravitational force density should be the contraction of the gravitational field and the energy-momentum tensor of matter. In the "Entwurf" theory, Einstein read off the expression for the gravitational field from the second term in eq. (15).[37] In the November 1915 theory, he used the second term of eq. (14) instead. Commenting on this switch in his first paper of November 1915, Einstein wrote:

> This conservation law [essentially eq. (15)] has led me in the past to look upon the quantities $[(1/2) g^{\tau\mu} g_{\mu\nu,\sigma}]$ as the natural expressions of the components of the gravitational field, even though the formulas of the absolute differential calculus suggest the Christoffel symbols […] instead. *This was a fateful prejudice.*[38]

35 Eq. (14) is equivalent to $T^{\alpha}_{\mu;\alpha} = 0$:

$$\sqrt{-g}\, T^{\alpha}_{\mu;\alpha} = \sqrt{-g}\left(T^{\alpha}_{\mu,\alpha} + \left\{ \begin{matrix} \alpha \\ \beta\alpha \end{matrix} \right\} T^{\beta}_{\mu} - \left\{ \begin{matrix} \beta \\ \mu\alpha \end{matrix} \right\} T^{\alpha}_{\beta} \right) = 0$$

Upon substitution of $\left\{ \begin{matrix} \alpha \\ \beta\alpha \end{matrix} \right\} = (\sqrt{-g})_{,\beta} / \sqrt{-g}$, this equation turns into

$$\sqrt{-g}\, T^{\alpha}_{\mu,\alpha} + (\sqrt{-g})_{,\alpha} T^{\alpha}_{\mu} - \left\{ \begin{matrix} \beta \\ \mu\alpha \end{matrix} \right\} \sqrt{-g}\, T^{\alpha}_{\beta} = 0 ,$$

which can be rewritten as eq. (14).

36 Using definition (2) of the Christoffel symbols, one can rewrite the second term of eq. (14) as

$$-\left\{ \begin{matrix} \beta \\ \mu\alpha \end{matrix} \right\} \mathrm{T}^{\alpha}_{\beta} = -\frac{1}{2} g^{\beta\rho} (g_{\rho\mu,\alpha} + g_{\rho\alpha,\mu} - g_{\mu\alpha,\rho}) \mathrm{T}^{\alpha}_{\beta} = -\frac{1}{2} \mathrm{T}^{\alpha\rho} (g_{\rho\mu,\alpha} - g_{\mu\alpha,\rho}) - \frac{1}{2} g^{\beta\rho} g_{\rho\alpha,\mu} \mathrm{T}^{\alpha}_{\beta} .$$

The first term of this last expression vanishes since it is the contraction of a quantity symmetric in the indices α and ρ and a quantity anti-symmetric in those same indices.

37 The motivation for this choice is explained in (Einstein 1914c, 1060, note 1). See also Einstein to Hans Thirring, 7 December 1917 (CPAE 8, Doc. 405, note 4).

38 "Diese Erhaltungsgleichung hat mich früher dazu verleitet, die Größen […] als den natürlichen Ausdruck für die Komponenten des Gravitationsfeldes anzusehen, obwohl es im Hinblick auf die Formeln des absoluten Differentialkalküls näher liegt, die Christoffelschen Symbole statt jener Größen einzuführen. Dies war ein verhängnisvolles Vorurteil" (Einstein 1915a, 782; our emphasis).

After the fourth paper of November 1915, he told Sommerfeld:

> *The key to this solution* was my realization that not [$g^{l\alpha} g_{\alpha i,m}$] but the related Christoffel symbols […] are to be regarded as the natural expression for the "components" of the gravitational field.[39]

To appreciate the full significance of these comments, one needs to see how the November tensor drops out of Einstein's variational formalism (see sec. 5 below). For now, we shall follow the treatment in (Einstein 1914c) and work with eq. (15) rather than with eq. (14).

In relativistic continuum mechanics, which is carefully tailored to electrodynamics, the theory for which it was first developed, a four-force density can be written as the four-divergence of a suitably chosen energy-momentum tensor.[40] So Einstein tried to write the gravitational force density in eq. (15) as the four-divergence of a suitably chosen gravitational energy-momentum (pseudo-)tensor density t_μ^α. If this can be done, energy-momentum conservation can be written as the vanishing of an ordinary divergence:

$$(\mathrm{T}_\mu^\lambda + \mathrm{t}_\mu^\lambda)_{,\lambda} = 0 . \tag{16}$$

Eq. (15) can indeed be written in this form, but the resulting expression for t_μ^α differs from expression (12) for t_μ^α found earlier. Einstein therefore had to add an extra condition to his theory that sets these two expressions equal to one another. As he discovered in October 1915, this same condition pops up in the analysis of the covariance properties of the theory.

Since $g^{\beta\rho} g_{\rho\alpha,\mu} = -g_\mu^{\beta\rho} g_{\rho\alpha}$, eq. (15) can also be written as:

$$\mathrm{T}_{\mu,\alpha}^\alpha + \frac{1}{2} g_\mu^{\alpha\beta} \mathrm{T}_{\alpha\beta} = 0 . \tag{17}$$

The second term has to be written in the form of $\mathrm{t}_{\mu,\lambda}^\lambda$. To this end, the field equations (11) are used to replace $\mathrm{T}_{\alpha\beta}$ by an expression in terms of the metric field and its derivatives:[41]

$$\frac{1}{2} g_\mu^{\alpha\beta} \mathrm{T}_{\alpha\beta} = \frac{1}{2\kappa} g_\mu^{\alpha\beta} \left(\frac{\partial Q}{\partial g^{\alpha\beta}} - \frac{\partial}{\partial x^\lambda} \left(\frac{\partial Q}{\partial g_\lambda^{\alpha\beta}} \right) \right) . \tag{18}$$

The right-hand side of this expression can indeed be written in the form $\mathrm{t}_{\mu,\lambda}^\lambda$ with:[42]

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39 "Den Schlüssel zu dieser Lösung lieferte mir die Erkenntnis, dass nicht […] sondern die damit verwandten Christoffel'schen Symbole […] als natürlichen Ausdruck für die „Komponente" des Gravitationsfeldes anzusehen ist." Einstein to Arnold Sommerfeld, 28 November 1915 (CPAE 8, Doc. 153). Our emphasis.

40 See, e.g., sec. 20 of Einstein's "Manuscript on the Special Theory of Relativity" (CPAE 4, Doc. 1).

41 This method for identifying the expression for the gravitational energy-momentum pseudo-tensor can already be found at several places in the Zurich Notebook (see, e.g., p. 19R and p. 24R, discussed in secs. 5.4.2 and 5.6.1, respectively, of "Commentary …" [this volume]). It was also used in (Einstein and Grossmann 1913, 15). Einstein had used a completely analogous method in his earlier theory for static fields (Einstein 1912, 456).

$$\mathrm{t}_\mu^\lambda \equiv \frac{1}{2\kappa}\left(\delta_\mu^\lambda Q - g_\mu^{\alpha\beta}\frac{\partial Q}{\partial g_\lambda^{\alpha\beta}}\right). \tag{19}$$

We introduce the more explicit notations $\mathrm{t}_\mu^\lambda(Q, \mathrm{cons})$ for t_μ^λ as defined in eq. (19) and $\mathrm{t}_\mu^\lambda(Q, \mathrm{source})$ for t_μ^λ as defined in eq. (12).[43]

Compatibility between these two definitions is assured if the quantity S_σ^ν, defined as (Einstein 1914c, 1075, eq. 76a),[44]

$$S_\sigma^\nu \equiv g_\alpha^{\beta\nu}\frac{\partial Q}{\partial g_\alpha^{\beta\sigma}} + g^{\alpha\nu}\frac{\partial Q}{\partial g^{\alpha\sigma}} + \frac{1}{2}\delta_\sigma^\nu Q - \frac{1}{2}g_\sigma^{\alpha\beta}\frac{\partial Q}{\partial g_\nu^{\alpha\beta}}, \tag{20}$$

vanishes. This quantity is equal to (κ times) the difference between the two definitions of t_μ^λ:

$$S_\sigma^\nu(Q) = \kappa\mathrm{t}_\sigma^\nu(Q, \mathrm{cons}) - \kappa\mathrm{t}_\sigma^\nu(Q, \mathrm{source}). \tag{21}$$

At the time of his review article on the "Entwurf" theory, Einstein thought that this condition uniquely determined $Q = \sqrt{-g}H$ to be the Lagrangian for the "Entwurf" field equations. As it turns out, S_σ^ν vanishes for *any* H that transforms as a scalar under general linear transformations (see eqs. (31)–(32) below).

Einstein imposed another set of conditions on the Lagrangian density Q which guarantee energy-momentum conservation. Taking the divergence of both sides of the field equations (13), one arrives at

$$-\frac{\partial^2}{\partial x^\lambda \partial x^\alpha}\left(g^{\nu\lambda}\frac{\partial Q}{\partial g_\alpha^{\mu\nu}}\right) = \kappa(\mathrm{T}_\mu^\lambda + \mathrm{t}_\mu^\lambda)_{,\lambda}. \tag{22}$$

The field equations thus imply energy-momentum conservation, if Q satisfies the condition (Einstein 1914c, 1077):

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42 Using that

$$\frac{\partial Q}{\partial x^\mu} = \frac{\partial Q}{\partial g^{\alpha\beta}}g_\mu^{\alpha\beta} + \frac{\partial Q}{\partial g_\lambda^{\nu\sigma}}g_{\lambda\mu}^{\alpha\beta},$$

with $g_{\lambda\mu}^{\alpha\beta} \equiv g^{\alpha\beta}{}_{,\lambda\mu}$, one can rewrite the first term on the right-hand side of eq. (18) as

$$g_\mu^{\alpha\beta}\frac{\partial Q}{\partial g^{\alpha\beta}} = \frac{\partial Q}{\partial x^\mu} - \frac{\partial Q}{\partial g_\lambda^{\nu\sigma}}g_{\lambda\mu}^{\alpha\beta}.$$

One thus arrives at

$$\frac{\partial Q}{\partial x^\mu} - g_{\mu\lambda}^{\alpha\beta}\frac{\partial Q}{\partial g_\lambda^{\nu\sigma}} - g_\mu^{\alpha\beta}\frac{\partial}{\partial x^\lambda}\left(\frac{\partial Q}{\partial g_\lambda^{\alpha\beta}}\right) = \frac{\partial}{\partial x^\lambda}\left(\delta_\mu^\lambda Q - g_\mu^{\alpha\beta}\frac{\partial Q}{\partial g_\lambda^{\alpha\beta}}\right),$$

from which eq. (19) follows.

43 The designations "source" and "cons[ervation]" refer to the fact that these two definitions are found from considerations of the source term of the field equations and considerations of energy-momentum conservation, respectively.

44 Our derivation of eq. (20) follows Einstein to H. A. Lorentz, 12 October 1915 (CPAE 8, Doc. 129). In his 1914 review article, Einstein derived this equation by substituting the left-hand side of eq. (11) for $\mathrm{T}_{\mu\nu}$ in both terms on the left-hand side of eq. (17).

$$B_\mu \equiv \frac{\partial^2}{\partial x^\rho \partial x^\sigma}\left(g^{\rho\tau}\frac{\partial Q}{\partial g_\sigma^{\mu\tau}}\right) = 0 . \tag{23}$$

Energy-momentum thus calls for two sets of conditions:

$$S_\sigma^\nu = 0 , \; B_\mu = 0 . \tag{24}$$

These same conditions, it turns out, also express the covariance properties of the field equations (see eq. (34) below).

3.3 ***Covariance properties.*** *The conditions the Lagrangian has to satisfy for the action to be invariant under a given coordinate transformation are determined. The assumption that the action is at least invariant under general linear transformations leads to the conditions $S_\sigma^\nu = 0$. Einstein did not explicitly write down these conditions in 1914, which explains why he thought that these conditions, which he did encounter in the context of energy-momentum conservation, could be used to determine the Lagrangian. The conditions for additional non-linear transformations leaving the action invariant are $B_\mu = 0$, which, as Einstein did recognize, were also the conditions guaranteeing energy-momentum conservation. In the "Entwurf" theory, these four conditions determine the class of what Einstein called "adapted coordinates." In general relativity, they turn into the generally-covariant contracted Bianchi identities.*

What are the transformations that leave the action $J = \int H\sqrt{-g}d\tau$ (with H an arbitrary function of $g^{\mu\nu}$ and $g_\alpha^{\mu\nu}$) invariant?[45] Consider an arbitrary infinitesimal coordinate transformation $x'^\mu = x^\mu + \Delta x^\mu$. The changes in $g^{\mu\nu}$ and $g_\alpha^{\mu\nu}$ under this transformation are given by[46,47]

$$\Delta g^{\mu\nu} \equiv g'^{\mu\nu}(x') - g^{\mu\nu}(x) = g^{\mu\alpha}\frac{\partial \Delta x^\nu}{\partial x^\alpha} + g^{\nu\alpha}\frac{\partial \Delta x^\mu}{\partial x^\alpha} , \tag{25}$$

$$\Delta g_\alpha^{\mu\nu} \equiv g'^{\mu\nu}_\alpha(x') - g_\alpha^{\mu\nu}(x) = \frac{\partial}{\partial x^\alpha}(\Delta g^{\mu\nu}) - \frac{\partial \Delta x^\beta}{\partial x^\alpha} g_\beta^{\mu\nu} . \tag{26}$$

The change in J is given by

$$\Delta J = J' - J = \int Q' d\tau' - \int Q d\tau . \tag{27}$$

The Jacobian $|\partial x'/\partial x|$ can be written as $1 + \partial\Delta x^\mu/\partial x^\mu$.[48] ΔJ can then be rewritten as

45 Ultimately, the question is under which transformations the field equations are invariant. Both in (Einstein and Grossmann 1914b) and in (Einstein 1914c, 1069–1071), Einstein argued that these are just the transformations under which the action is invariant. Levi-Civita's criticism was aimed at this part of Einstein's argument, which for our purposes is not important. Einstein and Grossmann (1914b, 219, note 2) credit Paul Bernays with the suggestion to use a variational formalism to investigate the covariance properties of the "Entwurf" field equations.

$$\Delta J = \int Q'\left(1 + \frac{\partial \Delta x^{\mu}}{\partial x^{\mu}}\right) d\tau - \int Q d\tau = \int Q \delta_{\sigma}^{\nu} \frac{\partial \Delta x^{\sigma}}{\partial x^{\nu}} d\tau + \int \Delta Q d\tau . \tag{28}$$

Since Q is a function of $g^{\mu\nu}$ and $g_{\alpha}^{\mu\nu}$, $\Delta Q \equiv Q' - Q$ is given by

$$\Delta Q = \frac{\partial Q}{\partial g^{\mu\nu}} \Delta g^{\mu\nu} + \frac{\partial Q}{\partial g_{\alpha}^{\mu\nu}} \Delta g_{\alpha}^{\mu\nu} .$$

Inserting eqs. (25)–(26) for $\Delta g^{\mu\nu}$ and $\Delta g_{\alpha}^{\mu\nu}$, one finds[49]

$$\Delta Q = \left\{ 2 g^{\alpha\nu} \frac{\partial Q}{\partial g^{\alpha\sigma}} + 2 g_{\alpha}^{\beta\nu} \frac{\partial Q}{\partial g_{\alpha}^{\beta\sigma}} - g_{\sigma}^{\alpha\beta} \frac{\partial Q}{\partial g_{\nu}^{\alpha\beta}} \right\} \frac{\partial \Delta x^{\sigma}}{\partial x^{\nu}} + 2 \frac{\partial Q}{\partial g_{\sigma}^{\mu\tau}} g^{\rho\tau} \frac{\partial^2 \Delta x^{\mu}}{\partial x^{\sigma} \partial x^{\rho}} . \tag{29}$$

Inserting this expression for ΔQ into expression (28) for ΔJ, one finds:

$$\begin{aligned} \frac{1}{2}\Delta J = & \int \left\{ g_{\alpha}^{\beta\nu} \frac{\partial Q}{\partial g_{\alpha}^{\beta\sigma}} + g^{\alpha\nu} \frac{\partial Q}{\partial g^{\alpha\sigma}} + \frac{1}{2} \delta_{\sigma}^{\nu} Q - \frac{1}{2} g_{\sigma}^{\alpha\beta} \frac{\partial Q}{\partial g_{\nu}^{\alpha\beta}} \right\} \frac{\partial \Delta x^{\sigma}}{\partial x^{\nu}} d\tau \\ & + \int g^{\rho\tau} \frac{\partial Q}{\partial g_{\sigma}^{\mu\tau}} \frac{\partial^2 \Delta x^{\mu}}{\partial x^{\rho} \partial x^{\sigma}} d\tau . \end{aligned} \tag{30}$$

The expression in curly brackets in the first integral is just the quantity S_{σ}^{ν} defined in eq. (20) in the course of the discussion of energy-momentum conservation. Eq. (30) can thus be written more compactly as

46 Einstein had an idiosyncratic way of computing the variations induced by coordinate transformations. Felix Klein, David Hilbert, Emmy Noether, and other mathematicians in or closely affiliated with Göttingen (such as Hermann Weyl in Zurich) used what is called the "Lie variation" of the metric tensor, defined as $\delta g_{\mu\nu} \equiv g'_{\mu\nu}(x) - g_{\mu\nu}(x)$. Commenting on (Weyl 1918), which he was reading in proof, Einstein wrote: "He [Weyl] derives the energy law for matter with the same variational trick that you used in the note that recently appeared [Klein 1917]" ("Den Energiesatz der Materie leitet er mit demselben Variations-Kunstgriff ab wie Sie in Ihrer neulich erschienenen Note." Einstein to Felix Klein, 24 March 1918 [CPAE 8, Doc. 492]). Two years earlier he had already noted that Lie variation and differentiation commute (Einstein to David Hilbert, 30 March 1916 [CPAE 8, Doc. 207]). This is not the case if one does the variation the way Einstein does (see eq. (26)). One would have expected Einstein to pick up on this quickly. After all, the crucial distinction between $g'_{\mu\nu}(x')$ and $g'_{\mu\nu}(x)$ was very familiar to him from the hole argument (see, e.g., sec. 4 in "What did Einstein know …". [this volume]). In fact, Einstein still had not fully assimilated the notion of Lie variation in late 1918, as can be inferred from his comments on (Klein 1918a): "At first I had some trouble understanding your equation (6) [involving Lie variation]. The point is that with your preferred way of doing variations $\delta(\partial g^{\mu\nu}/\partial x_{\sigma}) = \partial/\partial x_{\sigma}(\delta g^{\mu\nu})$" ("Anfänglich hatte ich etwas Mühe, Ihre Gleichung (6) zu begreifen. Der Witz ist eben, dass bei der von Ihnen bevorzugten Art zu varieren […] ist." Einstein to Felix Klein, 22 October 1918 [CPAE 8, Doc. 638]). Old habits die hard. In his lectures on general relativity in Berlin in 1919, Einstein still vacillated between his own way of doing the variations and that of Weyl and Klein (CPAE 7, Doc. 19, [pp. 13–17]).

$$\frac{1}{2}\Delta J = \int S_\sigma^\nu \frac{\partial \Delta x^\sigma}{\partial x^\nu} d\tau + \int g^{\rho\tau} \frac{\partial Q}{\partial g_\sigma^{\mu\tau}} \frac{\partial^2 \Delta x^\mu}{\partial x^\rho \partial x^\sigma} d\tau\,. \tag{31}$$

Einstein now focused his attention on functions H that transform as scalars under arbitrary linear transformations. This implies that the action J is invariant under linear transformations ($\sqrt{-g}d\tau$ is an invariant volume element). For linear transformations the second-order derivatives of Δx^μ all vanish, so the second integral in eq. (31) does not contribute to ΔJ. This means that the first integral must vanish identically for arbitrary values of the first-order derivatives of Δx^μ, i.e., that:

$$S_\sigma^\nu = 0\,. \tag{32}$$

Through partial integration, the second integral in eq. (31) can be rewritten as

$$\int \frac{\partial^2}{\partial x^\rho \partial x^\sigma}\left(g^{\rho\tau}\frac{\partial Q}{\partial g_\sigma^{\mu\tau}}\right)\Delta x^\mu d\tau$$

plus surface terms that can all be assumed to vanish. The integrand is the contraction of Δx^μ and an expression which is exactly equal to the quantity B_μ defined in eq. (23) in the context of the discussion of energy-momentum conservation:

$$B_\mu \equiv \frac{\partial^2}{\partial x^\rho \partial x^\sigma}\left(g^{\rho\tau}\frac{\partial Q}{\partial g_\sigma^{\mu\tau}}\right)\,.$$

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47 Eq, (25) for $\Delta g^{\mu\nu}$ follows from

$$\Delta g^{\mu\nu} = g'^{\mu\nu} - g^{\mu\nu} = \frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta} - g^{\mu\nu} = \left(\delta_\alpha^\mu + \frac{\partial \Delta x^\mu}{\partial x^\alpha}\right)\left(\delta_\beta^\nu + \frac{\partial \Delta x^\nu}{\partial x^\beta}\right) g^{\alpha\beta} - g^{\mu\nu}\,;$$

eq. (26) for $\Delta g_\sigma^{\mu\nu}$ from:

$$\begin{aligned}\Delta g_\sigma^{\mu\nu} &= g'^{\mu\nu}_\sigma - g_\sigma^{\mu\nu} = \frac{\partial x^\rho}{\partial x'^\sigma}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x'^\mu}{\partial x^\alpha}\frac{\partial x'^\nu}{\partial x^\beta} g^{\alpha\beta}\right) - g_\sigma^{\mu\nu} \\ &= \left(\delta_\sigma^\rho - \frac{\partial \Delta x^\rho}{\partial x^\sigma}\right)\frac{\partial}{\partial x^\rho}\left(\left(\delta_\alpha^\mu + \frac{\partial \Delta x^\mu}{\partial x^\alpha}\right)\left(\delta_\beta^\nu + \frac{\partial \Delta x^\nu}{\partial x^\beta}\right) g^{\alpha\beta}\right) - g_\sigma^{\mu\nu}.\end{aligned}$$

48 The Jacobian can be computed as follows:

$$\left|\frac{\partial x'}{\partial x}\right| = \varepsilon_{\mu..\nu}\frac{\partial x'^\mu}{\partial x^1}..\frac{\partial x'^\nu}{\partial x^4} = \varepsilon_{\mu..\nu}\delta_1^\mu..\delta_4^\nu + \varepsilon_{\mu..\nu}\frac{\partial x'^\mu}{\partial x^1}..\delta_4^\nu + \ldots + \ldots + \varepsilon_{\mu..\nu}\delta_1^\mu..\frac{\partial x'^\nu}{\partial x^4} = 1 + \frac{\partial \Delta x^\mu}{\partial x^\mu}\,.$$

49 Eq. (29) is found as follows:

$$\begin{aligned}\Delta Q &= \frac{\partial Q}{\partial g^{\mu\nu}}\left(g^{\mu\alpha}\frac{\partial \Delta x^\nu}{\partial x^\alpha} + g^{\nu\alpha}\frac{\partial \Delta x^\mu}{\partial x^\alpha}\right) + \frac{\partial Q}{\partial g_\alpha^{\mu\nu}}\left(\frac{\partial}{\partial x^\alpha}(\Delta g^{\mu\nu}) - \frac{\partial \Delta x^\beta}{\partial x^\alpha} g_\beta^{\mu\nu}\right) \\ &= 2\frac{\partial Q}{\partial g^{\mu\nu}} g^{\mu\alpha}\frac{\partial \Delta x^\nu}{\partial x^\alpha} + \frac{\partial Q}{\partial g_\alpha^{\mu\nu}}\left(\frac{\partial}{\partial x^\alpha}\left(2g^{\mu\beta}\frac{\partial \Delta x^\nu}{\partial x^\beta}\right) - \frac{\partial \Delta x^\beta}{\partial x^\alpha} g_\beta^{\mu\nu}\right) \\ &= 2\frac{\partial Q}{\partial g^{\mu\nu}} g^{\mu\alpha}\frac{\partial \Delta x^\nu}{\partial x^\alpha} + 2\frac{\partial Q}{\partial g_\alpha^{\mu\nu}} g_\alpha^{\mu\beta}\frac{\partial \Delta x^\nu}{\partial x^\beta} + 2\frac{\partial Q}{\partial g_\alpha^{\mu\nu}} g^{\mu\beta}\frac{\partial^2 \Delta x^\nu}{\partial x^\alpha \partial x^\beta} - \frac{\partial Q}{\partial g_\alpha^{\mu\nu}} g_\beta^{\mu\nu}\frac{\partial \Delta x^\beta}{\partial x^\alpha}.\end{aligned}$$

Grouping terms in $\partial \Delta x^\nu / \partial x^\alpha$ and in $\partial^2 \Delta x^\nu / \partial x^\alpha \partial x^\beta$ and relabeling indices, one arrives at eq. (29).

Assuming that condition (32) holds, one can thus rewrite eq. (31) as

$$\frac{1}{2}\Delta J = \int B_\mu \Delta x^\mu d\tau . \tag{33}$$

The upshot then is that the following two sets of conditions have to be satisfied for the action to be invariant under some coordinate transformation $x'^\mu = x^\mu + \Delta x^\mu$:

$$S_\sigma^\nu = 0 , \; B_\mu = 0 . \tag{34}$$

These conditions are the same as the conditions guaranteeing energy-momentum conservation that we found in eq. (24).

In the case of the four conditions $B_\mu = 0$, Einstein clearly recognized in his 1914 review article that they play this dual role (Einstein 1914c, 1076–1077). In the case of the conditions $S_\sigma^\nu = 0$, however, he did not. He only encountered these conditions in the context of energy-momentum conservation. He did not encounter them in his analysis of the covariance of the action J. Einstein started from

$$\Delta J = \int \Delta H \sqrt{-g} d\tau \tag{35}$$

rather than from eq. (27). He did not bother to write down the coefficients of $\partial \Delta x^\sigma / \partial x^\nu$ in ΔH as we did for ΔQ (see eq. (29)). He wrote:

> We now assume that H is invariant under linear transformations, i.e., that ΔH should vanish if the [$\partial^2 \Delta x^\mu / \partial x^\alpha \partial x^\sigma$] vanish. On this assumption we arrive at
>
> $$\left[\frac{1}{2}\Delta H = g^{\nu\alpha} \frac{\partial H}{\partial g_\sigma^{\mu\nu}} \frac{\partial^2 \Delta x^\mu}{\partial x^\alpha \partial x^\sigma} \right]$$
>
> With the help of [this equation and $\Delta(\sqrt{-g} d\tau) = 0$] one arrives at
>
> $$\left[\frac{1}{2}\Delta J = \int d\tau g^{\nu\alpha} \frac{\partial H \sqrt{-g}}{\partial g_\sigma^{\mu\nu}} \frac{\partial^2 \Delta x^\mu}{\partial x^\alpha \partial x^\sigma} \right]$$
>
> and from this through partial integration at
>
> $$\left[\frac{1}{2}\Delta J = \int d\tau (\Delta x^\mu B_\mu) + \text{surface terms} \right]$$ [50]

Using eq. (29) with H instead of Q and the invariance of $\sqrt{-g} d\tau$, one arrives at

$$\frac{1}{2}\Delta J = \int \left\{ g^{\alpha\nu} \frac{\partial H}{\partial g^{\alpha\sigma}} + g_\alpha^{\beta\nu} \frac{\partial H}{\partial g_\alpha^{\beta\sigma}} - \frac{1}{2} g_\sigma^{\alpha\beta} \frac{\partial H}{\partial g_\nu^{\alpha\beta}} \right\} \frac{\partial \Delta x^\sigma}{\partial x^\nu} \sqrt{-g} d\tau$$
$$+ \int g^{\rho\tau} \frac{\partial H}{\partial g_\sigma^{\mu\tau}} \frac{\partial^2 \Delta x^\mu}{\partial x^\rho \partial x^\sigma} \sqrt{-g} d\tau .$$

50 “Wir nehmen nun an daß H bezüglich linearer Transformationen eine Invariante sei; d. h. ΔH soll verschwinden, falls die […] verschwinden. Unter dieser Voraussetzung enthalten wir […]. Mit hilfe von […] erhält man […] und hieraus durch partielle Integration […]” (Einstein 1914c, 1069–1070).

The condition that the expression in curly brackets vanish guarantees that $\Delta J = 0$ for linear transformations. Comparing this expression to S_σ^ν, the coefficients of $\partial\Delta x^\sigma / \partial x^\nu$ in eq. (30), one sees that the condition read off of the expression for ΔJ above is equivalent but not identical to the conditions $S_\sigma^\nu = 0$. So even if Einstein actually did calculate the coefficients of $\partial\Delta x^\sigma / \partial x^\nu$ in ΔH and ΔJ, he would not have arrived at the conditions $S_\sigma^\nu = 0$ found in the context of energy-momentum conservation. Consequently, he would still not have realized that $S_\sigma^\nu = 0$ for any function that transforms as a scalar under general linear transformations and that these conditions therefore cannot be used to determine the Lagrangian.[51]

Almost a year went by before Einstein discovered his error. As he wrote to Lorentz in early October 1915:

> The invariant-theoretical method actually does not tell us more than the Hamiltonian principle when it comes to the determination of your function Q $(= H\sqrt{-g})$ [see Lorentz 1915, 763]. That I did not realize this last year is because I nonchalantly introduced the assumption on p. 1069 of my paper [Einstein 1914c] that H be an invariant under *linear* transformations.[52]

The conditions on Q coming from the "Hamiltonian principle" are presumably the ones coming from the requirement that expressions (13) and (20) for t_ν^μ are equal to one another. This is how Einstein derives the conditions $S_\sigma^\nu = 0$ in his letter to Lorentz. He then adds: "This is also the condition for $[Qd\tau]$ being an invariant under linear transformations" (ibid.).[53]

We now turn from the conditions $S_\sigma^\nu = 0$ to the conditions $B_\mu = 0$. These are the conditions for "coordinates adapted to the gravitational field."[54] or "adapted coordinates" for short. If some metric field $g^{\mu\nu}$ expressed in coordinates x^μ satisfies these conditions, the coordinates are called adapted to that field. Transformations from one adapted coordinate system to another are called "justified."[55] Such transformations are not mappings of the form $x^\mu \to x'^\mu$, like ordinary coordinate transformations, but mappings of the form $(x^\mu, g^{\mu\nu}(x)) \to (x'^\mu, g'^{\mu\nu}(x'))$. Because of their

51 In (Einstein 1916c, 1113–1115), the conditions $S_\sigma^\nu = 0$ and $B_\mu = 0$ are derived in yet another way (see sec. 9, eqs. (118)–(123)).

52 "Thatsächlich liefert die invariantentheoretische Methode nicht mehr als das Hamilton'sche Prinzip wenn es sich um die Bestimmung der Ihrer Funktion Q $(= H\sqrt{-g})$ handelt. Dass ich dies letztes Jahr nicht merkte liegt daran, dass ich auf Seite 1069 meiner Abhandlung leichtsinnig die Voraussetzung einführte, H sei eine Invariante bezüglich linearer Transformationen." Einstein to H. A. Lorentz, October 12, 1915 (CPAE 8, Doc. 129). The function Q was introduced in (Lorentz 1915, 763).

53 "Dies ist gleichzeitig die Bedingung dafür, dass $[Qd\tau]$ eine Invariante bezüglich linearer Substitutionen ist."

54 "dem Gravitationsfeld angepaßte Koordinatensysteme" (Einstein 1914c, 1070). In (Einstein and Grossmann 1914b, 221), such coordinates are called "'adapted' to the manifold" ("der Mannigfaltigkeit „*angepaßte*“").

55 "„*berechtigte*“" (Einstein and Grossmann 1914b, 221).

dependence on the metric, Einstein, at Ehrenfest's suggestion, called them "non-autonomous" transformations at one point.[56]

Einstein looked upon the conditions $B_\mu = 0$ for adapted coordinates as the coordinate restriction with which the "Entwurf" field equations could be extracted from unknown generally-covariant equations.[57] He must have been pleased to see that these coordinate restrictions follow from energy-momentum conservation. In March 1914, Einstein wrote a letter to Besso reporting on the results that would be published a few months later in (Einstein and Grossmann 1914b). He showed how the conditions $B_\mu = 0$ follow from the field equations and energy-momentum conservation (cf. eqs. (22)–(23)). The "Entwurf" field equations, he told Besso, "*hold in every frame of reference adapted to this condition.*"[58] Einstein claimed that this class of reference frames included all sorts of accelerated frames, including the important case of a rotating frame.[59] This is not true,[60] but that does not matter for our purposes. What is interesting for our story is the following passage from the draft of Besso's reply:

> You already had the fundamental insight that the conservation laws represent the condition for positing an admissible coordinate system; but it did not appear to be ruled out that a restriction to Lorentz transformations was thereby essentially already given, so that nothing particularly interesting epistemologically comes out of it. Now everything is fundamentally completely satisfactory.[61]

In the first sentence Besso is referring to Einstein's argument of late 1913 which seemed to show that energy-momentum conservation limits the covariance of the

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56 "„unselbständige"." Einstein to H. A. Lorentz, 14 August 1913 (CPAE 5, Doc. 467). Non-autonomous transformations play an important role in the Zurich Notebook (see sec. 4.3 of "Commentary …" [this volume]).

57 This fits with Einstein's general attitude towards general covariance at the time. In (Einstein 1914a, 177–178), he wrote: "When one has equations relating certain quantities that only hold in certain coordinate systems, one has to distinguish between two cases: 1. There are generally-covariant equations corresponding to the equations […]; 2. There are no generally-covariant equations that can be found on the basis of the equations given for a particular choice of reference frame. In case 2, the equations do not tell us anything about the things represented by these quantities; they only restrict the choice of reference frame. If the equations tell us anything at all about the things represented by these quantities, we are always dealing with case 1" ("Wenn Gleichungen zwischen irgendwelchen Größen gegeben sind, die nur bei spezieller Wahl des Koordinatensystems gültig sind, so sind zwei Fälle zu unterscheiden: 1. Es entsprechen den Gleichungen allgemein kovariante Gleichungen […]; 2. es gibt keine allgemein kovarianten Gleichungen, die aus den für spezielle Wahl des Bezugssystems gegebenen Gleichungen gefolgert werden können. Im Falle 2 sagen die Gleichungen über die durch die Größen dargestellten Dinge gar nichts aus; sie beschränken nur die Wahl des Bezugssystems. Sagen die Gleichungen über die durch die Größen dargestellten Dinge überhaupt etwas aus, so liegt stets der Fall 1 vor […]."). Similarly, he told Ehrenfest: "Grossmann wrote to me that he has now also been able to derive the gravitational [field] equations from the theory of general covariants. That would be a neat addition to our investigation [i.e., Einstein and Grossmann 1914b]" ("Grossmann schrieb mir, dass es ihm nun auch gelinge, die Gravitationsgleichungen aus der allgemeinen Kovariantentheorie abzuleiten. Es wäre dies eine hübsche Ergänzung zu unserer Untersuchung." Einstein to Paul Ehrenfest, before 10 April 1914 [CPAE 8, Doc. 2])

"Entwurf" field equations to linear transformations. Einstein had touted this specious result in several places. To Paul Ehrenfest, for instance, he wrote in November 1913, referring both to the argument from energy-momentum conservation and to the 'hole argument':

> The gravitation affair has been resolved to my *full satisfaction* (namely the circumstance that the equations of the gravitational field are only invariant under linear transformations[)]. It turns out that one can prove that *generally-covariant* equations that *fully* determine the field on the basis of the matter [energy-momentum] tensor cannot exist at all. What can be more beautiful than that the necessary specialization follows from the conservation laws?[62]

The argument that energy-momentum conservation restricts the covariance of the field equations to linear transformations evaporated early in 1914. But the triumphant last line of this letter to Ehrenfest can also be applied to the argument leading to the condition $B_\mu = 0$ that took its place: "What can be more beautiful than that the necessary specialization follows from the conservation laws?"

Neither in (Einstein and Grossmann 1914b) nor in (Einstein 1914c) do we find statements drawing attention to the close connection between covariance of the field equations and energy-momentum conservation. Einstein probably did emphasize this connection though in his Wolfskehl lectures in Göttingen in the summer of 1915.[63] Afterwards, in two letters to his friend Heinrich Zangger,[64] Einstein expressed his

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58 "… *für jedes Bezugssystem gelten, welches dieser Bedingung angepasst ist.*" (Einstein to Michele Besso, ca. 10 March 1914 [CPAE 5, Doc. 514]). Levi-Civita constructed a counter-example to Einstein's claim (Tullio Levi-Civita to Einstein, 28 March 1915 [CPAE 8, Doc. 67]). In our notation, Levi-Civita found a non-autonomous transformation $(x^\mu, g^{\mu\nu}(x)) \to (x'^\mu, g'^{\mu\nu}(x'))$ satisfying the condition for justified transformations between adapted coordinates (i.e., $B_\mu(g^{\mu\nu}(x)) = B_\mu(g'^{\mu\nu}(x')) = 0$), under which the "Entwurf" field equations were nonetheless *not* invariant (i.e., $g^{\mu\nu}(x)$ is a solution but $g'^{\mu\nu}(x')$ is not). In Levi-Civita's example, $g^{\mu\nu}(x) = \eta^{\mu\nu} = \mathrm{diag}(-1, -1, -1, 1)$.

59 This claim was based on a general argument given in Einstein to H.A. Lorentz, 23 January 1915 (CPAE 8, Doc. 47).

60 For the Minkowski metric in rotating coordinates, $B_\mu \neq 0$ (Janssen 1999, 150–151, note 47)

61 "Du hattest schon principiell eingesehen, dass die Erhaltungssätze die Bedingung für die Aufstellung eines zulässigen Coordinatensystems darstellen; aber es schien nicht ausgeschlossen, dass schon dadurch, im Wesentlichen, die Beschränkung auf die Lorentztransformationen gegeben sei, so dass nichts erkenntnistheoretisch besonders interessantes herauskam. Nun ist alles principiell vollkommen befriedigend." Michele Besso to Einstein, draft, 20 March 1914 (CPAE 5, Doc. 516). For further discussion of this letter, see sec. 2.2 of "What did Einstein know …" (this volume).

62 "Die Gravitationsaffäre hat sich zu meiner *vollen Befriedigung* aufgeklärt (der Umstand nämlich, dass die Gleichungen des Gr. Feldes nur linearen Transformationen gegenüber kovariant sind. Es lässt sich nämlich beweisen, dass *allgemein kovariante* Gleichungen, die das Feld aus dem materiellen Tensor vollständig bestimmen, überhaupt nicht existieren können. Was kann es schöneres geben, als dies, dass jene nötige Spezialisierung aus den Erhaltungssätzen fliesst?" Einstein to Paul Ehrenfest, before 7 November 1913 (Vol. 5, Doc. 481).

63 Notes taken by an unknown auditor present at (some of) these lectures were found by Leo Corry and are published in Appendix B of CPAE 6. These notes, however, do not touch on the field equations, nor on energy-momentum conservation.

satisfaction that he had been able to convince the Göttingen mathematicians, and David Hilbert in particular, of his "Entwurf" theory. In November 1915, Einstein found himself in a race against Hilbert to find field equations to replace the discarded "Entwurf" equations. Page proofs of (Hilbert 1915),[65] the final version of which would not be published until late March 1916, show that the theory originally proposed by Hilbert has a structure that is remarkably similar to that of the "Entwurf" theory as presented in (Einstein and Grossmann 1914b) and (Einstein 1914c). In these page proofs, Hilbert introduces field equations that are invariant under arbitrary transformations of the coordinates—or, as Hilbert calls them, "world parameters" ("Weltparameter"). He then rehearses what is essentially Einstein's 'hole argument' to argue that these "world parameters" need to be restricted to what he calls "space-time coordinates" ("Raum-Zeitkoordinaten"). Such space-time coordinates are defined as those world parameters for which a condition called the "energy theorem" ("Energiesatz") holds.[66] It is probably because of these similarities between Hilbert's original theory and the "Entwurf" theory, that Einstein accused Hilbert of "nostrification" ("Nostrifikation") in another letter to Zangger.[67] This episode is interesting for our purposes since it provides circumstantial evidence for our conjecture that Einstein did mention in his Wolfskehl lectures that it should be possible to extract the "Entwurf" field equations from (unknown) generally-covariant equations by imposing the coordinate restriction $B_\mu = 0$ given by the demands of energy-momentum conservation.

4. THE MAXWELLIAN "ENTWURF" LAGRANGIAN

We feed the gravitational part of the Lagrangian for the "Entwurf" field equations, modelled on the Lagrangian for the free Maxwell field, into the general formalism of (Einstein 1914c) and derive the field equations, the expression for the gravitational energy-momentum pseudo-tensor, and the condition for "adapted coordinates" that determines for each solution what other coordinate representations of the solution are also allowed by the "Entwurf" field equations.

One arrives at the gravitational part of the Lagrangian $Q = H\sqrt{-g}$ for the "Entwurf"

64 Einstein to Heinrich Zangger, 7 July 1915 (CPAE 8, Doc. 94) and between 24 July and 7 August 1915 (CPAE 8, Doc. 101).

65 These page proofs are located at the Niedersächsische Staats- und Universitätsbibliothek in Göttingen (Cod. Ms. D. Hilbert 634), where they were discovered by Leo Corry. For discussion, see (Corry et al. 1997), (Renn and Stachel 1999), and (Sauer 1999).

66 This restriction is stated in "Axiom III (Axiom of space and time)" ("Axiom III (Axiom von Raum und Zeit")) in the page proofs. This axiom is also mentioned in David Hilbert to Einstein, 13 November 1915 (CPAE 8, Doc. 140). It no longer occurs in (Hilbert 1915).

67 Einstein to Heinrich Zangger, 26 November 1915 (CPAE 8, Doc. 152). From Einstein to David Hilbert, 18 November 1915 (CPAE 8, Doc. 148) it can be inferred that Einstein saw a manuscript with an early version of the theory that would eventually be published in (Hilbert 1915).

field equations through the following choice for the function H (Einstein 1914c, 1076, note 1):

$$H = -g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\,, \tag{36}$$

where $\Gamma^{\alpha}_{\beta\mu}$ are the components of the gravitational field, defined as (Einstein 1914, p. 1077, eq. 81a):[68]

$$\Gamma^{\alpha}_{\beta\mu} \equiv \frac{1}{2}g^{\alpha\rho}g_{\rho\beta,\mu}\,. \tag{37}$$

The Lagrangian is modelled on the Lagrangian $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ for the free Maxwell field. Since H is a scalar under linear transformations, the conditions $S^{\nu}_{\sigma} = 0$ are satisfied (see eqs. (31)–(32)).

The definition of the components of the gravitational field is suggested by the energy-momentum balance equation, written in the form of eq. (15), which with the help of eq. (37) can be rewritten as

$$\mathrm{T}^{\alpha}_{\mu,\alpha} - \Gamma^{\beta}_{\alpha\mu}\mathrm{T}^{\alpha}_{\beta} = 0\,.$$

The second term represents the gravitational force density and has the same form as the Lorentz force density, $f_{\mu} = F_{\mu\nu}j^{\nu}$. It is the contraction of the field $\Gamma^{\alpha}_{\beta\mu}$ and its source $\mathrm{T}^{\alpha}_{\beta}$.

The quantities $\Gamma^{\alpha}_{\beta\mu}$ in eq. (37) are truncated versions of the Christoffel symbols[69]

$$\left\{ \begin{matrix} \alpha \\ \beta\mu \end{matrix} \right\} = \frac{1}{2}g^{\alpha\rho}(g_{\rho\beta,\mu} + g_{\rho\mu,\beta} - g_{\beta\mu,\rho})\,.$$

Note that, unlike the Christoffel symbols, the $\Gamma^{\alpha}_{\beta\mu}$-s are not symmetric in their lower indices.

We derive the left-hand side of the "Entwurf" field equations from the action principle $\delta J = 0$. The gravitational part of the action is (see eq. (9))

$$J = \int Q d\tau\,,$$

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68 This expression actually differs by a factor $1/2$ from the expression that leads to the "Entwurf" field equations as given in Einstein's publications prior to (Einstein 1914c). Substituting eq. (37) for $\Gamma^{\rho}_{\mu\tau}$ into eq. (36), one can rewrite the function H as:

$$H = -g^{\mu\nu}\left(\frac{1}{2}g^{\alpha\sigma}g_{\sigma\beta,\mu}\right)\left(\frac{1}{2}g^{\beta\tau}g_{\tau\alpha,\nu}\right).$$

Since $g^{\alpha\sigma}g^{\beta\tau}g_{\tau\alpha,\nu} = -g^{\sigma\beta}_{\nu}$, this expression can also be written as (Einstein 1914, 1076, eq. 78):

$$H = \frac{1}{4}g^{\mu\nu}g_{\sigma\beta,\mu}g^{\sigma\beta}_{\nu}\,.$$

To recover the "Entwurf" field equations, the factor $1/4$ should be replaced by $1/2$ (see Einstein and Grossmann 1914, 219, eq.Va, and note 72 below). In other words, the function H should be defined as $H = -2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$.

with $Q = \sqrt{-g}H$. There are two contributions to δQ:

$$\delta Q = \sqrt{-g}\delta H + (\delta\sqrt{-g})H\,. \tag{38}$$

Since $\delta g = -g g_{\alpha\beta}\delta g^{\alpha\beta}$ (see Einstein 1914c, 1051, eqs. 32–34; Einstein 1916a, 796, eq. 29),

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}\,. \tag{39}$$

Inserting this expression and eq. (36) for H into the second contribution to δQ in eq. (38), one arrives at:

$$(\delta\sqrt{-g})H = \left(\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\right)\delta g^{\mu\nu}\,. \tag{40}$$

Variation of H in the first contribution to δQ in eq. (38) gives:

$$\delta H = \delta(-g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}) = -\delta g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} - 2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\delta\Gamma^{\beta}_{\alpha\nu}\,.$$

For $\delta\Gamma^{\beta}_{\alpha\nu}$ one finds:[70]

$$\delta\Gamma^{\beta}_{\alpha\nu} = -\frac{1}{2}g_{\alpha\lambda}\delta g^{\beta\lambda}_{\nu} - g_{\alpha\rho}\Gamma^{\beta}_{\sigma\nu}\delta g^{\rho\sigma}\,.$$

It follows that:[71]

$$-2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\delta\Gamma^{\beta}_{\alpha\nu} = g^{\alpha\beta}g_{\rho\nu}\Gamma^{\rho}_{\mu\beta}\delta g^{\mu\nu}_{\alpha} + 2g^{\rho\tau}g_{\alpha\mu}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\tau}\delta g^{\mu\nu}\,.$$

Substituting this into the expression for δH above and collecting terms with $\delta g^{\mu\nu}$

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69 On p. 23L of the Zurich Notebook, Einstein had tried to extract field equations from the November tensor by truncating the Christoffel symbols in a similar fashion. Introducing the quantities

$$\vartheta_{il\alpha} \equiv \frac{1}{2}(g_{il,\alpha} + g_{l\alpha,i} + g_{\alpha i,l})$$

Einstein could write the Christoffel symbols as

$$\begin{Bmatrix} k \\ il \end{Bmatrix} = g^{k\alpha}(\vartheta_{il\alpha} - g_{il,\alpha})\,.$$

Inserting this expression into the November tensor,

$$T^{x}_{il} = \frac{\partial}{\partial x^k}\begin{Bmatrix} k \\ il \end{Bmatrix} - \begin{Bmatrix} \lambda \\ ik \end{Bmatrix}\begin{Bmatrix} k \\ \lambda l \end{Bmatrix}$$

(see eq. (5)), and eliminating all terms involving $\vartheta_{il\alpha}$ with the help of the appropriate coordinate restriction, Einstein arrived at the following candidate for the left-hand side of the field equations

$$-\frac{\partial}{\partial x^k}(g^{k\alpha}g_{il,\alpha}) - (g^{\lambda\alpha}g_{ik,\alpha})(g^{k\beta}g_{\lambda l,\beta})\,.$$

As Einstein realized, this expression can be obtained in one fell swoop by setting $\vartheta_{il\alpha} = 0$ and substituting $-g^{k\alpha}g_{il,\alpha}$ for the Christoffel symbols in the November tensor. Like all other candidates extracted from the Riemann tensor in the Zurich Notebook, this candidate was rejected because the necessary coordinate restriction turned out to be too restrictive. For a more detailed analysis, see sec. 5.5.4 of “Commentary …” (this volume).

and $\delta g^{\mu\nu}_{\alpha}$, one finds:

$$\delta H = (2g^{\rho\tau}g_{\alpha\mu}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\tau} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu})\delta g^{\mu\nu} + (g^{\alpha\beta}g_{\rho\nu}\Gamma^{\rho}_{\mu\beta})\delta g^{\mu\nu}_{\alpha}. \tag{41}$$

Inserting eqs. (40) and (41) into eq. (38), one finds:

$$\delta Q = \sqrt{-g}\Big[\Big(2g^{\rho\tau}g_{\alpha\mu}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\tau} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\Big)\delta g^{\mu\nu} + (g^{\alpha\beta}g_{\rho\nu}\Gamma^{\rho}_{\mu\beta})\delta g^{\mu\nu}_{\alpha}\Big].$$

Comparison of this expression with $\delta Q = \frac{\partial Q}{\partial g^{\mu\nu}}\delta g^{\mu\nu} + \frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\delta g^{\mu\nu}_{\alpha}$ gives

$$\frac{\partial Q}{\partial g^{\mu\nu}} = \sqrt{-g}\Big(2g^{\rho\tau}g_{\alpha\mu}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\tau} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\Big), \tag{42}$$

$$\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}} = \sqrt{-g}g^{\alpha\beta}g_{\rho\nu}\Gamma^{\rho}_{\mu\beta}. \tag{43}$$

Inserting equations (42)–(43) into the general form of the field equations,

$$\frac{\partial}{\partial x^{\alpha}}\Big(\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\Big) - \frac{\partial Q}{\partial g^{\mu\nu}} = -\kappa T_{\mu\nu} \tag{44}$$

(see eq. (11)), one can write the “Entwurf” field equations as:[72]

$$\frac{\partial}{\partial x^{\alpha}}(\sqrt{-g}g^{\alpha\beta}g_{\rho\nu}\Gamma^{\rho}_{\mu\beta}) - \sqrt{-g}\Big(2g^{\rho\tau}g_{\alpha\mu}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\nu\tau} - \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\Big) = -\kappa T_{\mu\nu}. \tag{45}$$

The “Entwurf” field equations can be written in a more compact form (Einstein 1914c, 1077). Instead of using eq. (44), one can use the field equations in the general form

70 Using eq. (37) for $\Gamma^{\beta}_{\alpha\nu}$, one finds:

$$\delta\Gamma^{\beta}_{\alpha\nu} = \frac{1}{2}\delta(g^{\beta\lambda}g_{\lambda\alpha,\nu}) = -\frac{1}{2}\delta(g^{\beta\lambda}_{\nu}g_{\lambda\alpha}) = -\frac{1}{2}\delta g^{\beta\lambda}_{\nu}g_{\lambda\alpha} - \frac{1}{2}g^{\beta\lambda}_{\nu}\delta g_{\lambda\alpha}.$$

Since $\delta g_{\alpha\lambda} = -g_{\alpha\rho}g_{\lambda\sigma}\delta g^{\rho\sigma}$, the last term can be rewritten as:

$$-\frac{1}{2}g^{\beta\lambda}_{\nu}\delta g_{\alpha\lambda} = \frac{1}{2}g^{\beta\lambda}_{\nu}g_{\alpha\rho}g_{\lambda\sigma}\delta g^{\rho\sigma} = -\frac{1}{2}g^{\beta\lambda}g_{\lambda\sigma,\nu}g_{\alpha\rho}\delta g^{\rho\sigma} = -g_{\alpha\rho}\Gamma^{\beta}_{\sigma\nu}\delta g^{\rho\sigma}.$$

71 This follows from

$$-2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\delta\Gamma^{\beta}_{\alpha\nu} = g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}(g_{\alpha\lambda}\delta g^{\beta\lambda}_{\nu}) + 2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}(g_{\alpha\rho}\Gamma^{\beta}_{\sigma\nu}\delta g^{\rho\sigma})$$

after relabeling indices.

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\lambda}\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}}\right) = -\kappa(\mathrm{T}^{\lambda}_{\mu} + \mathrm{t}^{\lambda}_{\mu}) \tag{46}$$

(see eq. (13)), along with eq. (19) for the gravitational energy-momentum tensor[73]

$$\kappa \mathrm{t}^{\lambda}_{\mu} = \frac{1}{2}\left(\delta^{\lambda}_{\mu}Q - g^{\alpha\beta}_{\mu}\frac{\partial Q}{\partial g^{\alpha\beta}_{\lambda}}\right). \tag{47}$$

Using expression (36) for H in $Q = H\sqrt{-g}$, one recovers the "Entwurf" field equations as given by Einstein. From eq. (43) it follows that eq. (46) in this case is:[74]

$$\frac{\partial}{\partial x^{\alpha}}(\sqrt{-g}g^{\alpha\beta}\Gamma^{\lambda}_{\mu\beta}) = -\kappa(\mathrm{T}^{\lambda}_{\mu} + \mathrm{t}^{\lambda}_{\mu}). \tag{48}$$

From eqs. (36) and (43) it follows that eq. (47) in this case is:

$$\kappa \mathrm{t}^{\lambda}_{\mu} = \frac{1}{2}(\delta^{\lambda}_{\mu}\sqrt{-g}[-g^{\rho\tau}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\tau}] - g^{\alpha\beta}_{\mu}[\sqrt{-g}g^{\lambda\rho}g_{\tau\beta}\Gamma^{\tau}_{\alpha\rho}]). \tag{49}$$

Simplifying this expression, one finds[75]

72 Expressing $\Gamma^{\alpha}_{\mu\nu}$ in terms of $g_{\mu\nu}$ and multiplying the left-hand side of eq. (45) by 2 to correct for the error in eq. (36) for H (see note 68), one recovers the "Entwurf" field equation as originally given in (Einstein and Grossmann 1913). The first two terms on the left-hand side of eq. (45) can be written as:

$$\frac{\partial}{\partial x^{\alpha}}\left(\sqrt{-g}g^{\alpha\beta}g_{\rho\nu}\left[\frac{1}{2}g^{\rho\lambda}g_{\lambda\mu,\beta}\right]\right) - 2\sqrt{-g}g^{\rho\tau}g_{\alpha\mu}\left(\frac{1}{2}g^{\alpha\sigma}g_{\sigma\beta,\rho}\right)\left(\frac{1}{2}g^{\beta\lambda}g_{\lambda\nu,\tau}\right)$$
$$= \frac{1}{2}\sqrt{-g}\left\{\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^{\alpha}}(\sqrt{-g}g^{\alpha\beta}g_{\mu\nu,\beta}) - g^{\alpha\beta}g^{\tau\rho}g_{\mu\tau,\alpha}g_{\nu\rho,\beta}\right\}.$$

The expression in curly brackets is the quantity $D_{\mu\nu}(g)$ defined in (Einstein and Grossmann 1913, 16, eq. 16). The last two terms on the left-hand side eq. (45) can likewise be written as:

$$\sqrt{-g}\left(\left[\frac{1}{2}g^{\alpha\rho}g_{\rho\beta,\mu}\right]\left[\frac{1}{2}g^{\beta\sigma}g_{\sigma\alpha,\nu}\right] - \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\left[\frac{1}{2}g^{\alpha\tau}g_{\tau\beta,\rho}\right]\left[\frac{1}{2}g^{\beta\lambda}g_{\lambda\alpha,\sigma}\right]\right)$$
$$= -\frac{1}{2}\sqrt{-g}\left\{\frac{1}{2}\left[g^{\alpha\beta}_{\mu}g_{\alpha\beta,\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}g^{\tau\rho}_{\alpha}g_{\tau\rho,\beta}\right]\right\}.$$

The expression in square brackets is the quantity $-2\kappa t_{\mu\nu}$ in (Einstein and Grossmann 1913, 16, eq. 14). The left-hand side of eq. (45) can thus be rewritten as:

$$\frac{1}{2}\sqrt{-g}\{D_{\mu\nu}(g) + \kappa t_{\mu\nu}\}.$$

Omitting the erroneous factor $1/2$ and dividing by $\sqrt{-g}$, one sees that eq. (45) can be rewritten as

$$-D_{\mu\nu}(g) = \kappa(t_{\mu\nu} + T_{\mu\nu}),$$

which is just (Einstein and Grossmann 1913, 17, eq. 21).

73 This expression is simpler than expression (12), which depends both on $\partial Q/\partial g^{\alpha\beta}_{\lambda}$ and on $\partial Q/\partial g^{\mu\nu}$. Since $S^{\nu}_{\sigma} = 0$, expressions (12) and (19) are equivalent (see eq. (19)–(21)).

$$\kappa t_{\mu}^{\lambda} = \sqrt{-g}\left(g^{\lambda\rho}\Gamma_{\tau\mu}^{\alpha}\Gamma_{\alpha\rho}^{\tau} - \frac{1}{2}\delta_{\mu}^{\lambda}g^{\rho\tau}\Gamma_{\beta\rho}^{\alpha}\Gamma_{\alpha\tau}^{\beta}\right). \tag{50}$$

This is indeed the expression for the gravitational energy-momentum pseudo-tensor as given in (Einstein 1914c, 1077, eq. 81b). And with this expression for t_{μ}^{λ}, the field equations (48) are indeed the "Entwurf" field equations as given in (Einstein 1914c, 1077, eq. 81).[76]

From eq. (48) it follows that the conditions $B_{\mu} = 0$—playing the dual role of restricting the coordinates (see eq. (33)) and guaranteeing the vanishing of the divergence of $t_{\mu}^{\lambda} + T_{\mu}^{\lambda}$ (see eqs. (22)–(23))—take on the specific form:

$$B_{\mu} = \frac{\partial^2}{\partial x^{\alpha}\partial x^{\lambda}}(\sqrt{-g}g^{\alpha\beta}\Gamma_{\mu\beta}^{\lambda}) = 0. \tag{51}$$

Commenting on eq. (37) for the gravitational field, eq. (48), the "Entwurf" field equations, and eq. (50) for the gravitational energy-momentum pseudo-tensor, Einstein wrote

> Despite its complexity, the system of equations admits of a simple physical interpretation. The left-hand side [of eq. (48)] expresses a kind of divergence of the gravitational field [eq. (37)]. This [divergence] is—as the right-hand side shows—determined by the components of the total energy tensor. What is very important is the result that the energy tensor of the gravitational field [eq. (50)] acts as a source of the field in the same way as the energy tensor of matter.[77]

On the preceding page, Einstein boasted that his new derivation of the "Entwurf" field equations is essentially free from physical considerations. After showing that the expression for H in eq. (36) satisfies the conditions $S_{\sigma}^{\nu} = 0$—as would any other expression transforming as a scalar under linear transformations—he wrote:

> We have now in a completely formal manner, i.e., without direct use of our physical knowledge about gravity, arrived at very definite field equations.[78]

Even if we forget for a moment that the uniqueness argument immediately preceding

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74 In detail: $\frac{\partial}{\partial x^{\alpha}}(g^{\nu\lambda}[\sqrt{-g}g^{\alpha\beta}g_{\rho\nu}\Gamma_{\mu\beta}^{\rho}]) = \frac{\partial}{\partial x^{\alpha}}(\sqrt{-g}g^{\alpha\beta}\delta_{\rho}^{\lambda}\Gamma_{\mu\beta}^{\rho}) = \frac{\partial}{\partial x^{\alpha}}(\sqrt{-g}g^{\alpha\beta}\Gamma_{\mu\beta}^{\lambda})$

75 The last term in eq. (49) can be rewritten as:

$$-\frac{1}{2}\sqrt{-g}g_{\mu}^{\alpha\beta}g^{\lambda\rho}g_{\tau\beta}\Gamma_{\alpha\rho}^{\tau} = \frac{1}{2}\sqrt{-g}g^{\alpha\beta}g^{\lambda\rho}g_{\tau\beta,\mu}\Gamma_{\alpha\rho}^{\tau} = \sqrt{-g}g^{\lambda\rho}\Gamma_{\tau\mu}^{\alpha}\Gamma_{\alpha\rho}^{\tau},$$

where in the last step eq. (37) was used.

76 To obtain the "Entwurf" field equations of Einstein's earlier publications, one needs to multiply both the left-hand side of eq. (48) and the right-hand sides of eqs. (49)–(50) by 2 (see notes 68 and 72).

77 "Das Gleichungssystem […] läßt trotz seiner Kompliziertheit eine einfache physikalische Interpretation zu. Die linke Seite drückt eine Art Divergenz des Gravitationsfeldes aus. Diese wird—wie die rechte Seite zeigt—bedingt durch die Komponente des totalen Energietensors. Sehr wichtig ist dabei das Ergebnis, daß der Energietensor des Gravitationsfeldes selbst in gleicher Weise felderregend wirksam ist wie der Energietensor der Materie" (Einstein 1914c, 1077).

it is hogwash, this statement is patently false. The derivation of the "Entwurf" field equations in (Einstein 1914c, part D) may be more formal than earlier derivations, but it still relies heavily on physical considerations. The function H giving the Lagrangian is modelled on the Lagrangian for the free Maxwell field. It is assumed to depend only on first-order derivatives of the metric because the Poisson equation of Newtonian theory suggests that the gravitational field equations do not contain anything higher than second-order derivatives of the metric (Einstein and Grossmann 1913, 11). The conditions $S_{\sigma}^{\nu} = 0$ that supposedly determine H uniquely are derived from the energy-momentum balance law of matter in a gravitational field (see sec. 3.2). Moreover, H only gives the gravitational part of the field equations. The matter part, $\mathrm{T}_{\mu}^{\lambda}$, is inserted on the basis of physical considerations. The same is true for the way in which the gravitational part of the field equations is split into a term with the divergence of the gravitational field and a term with the gravitational energy-momentum pseudo-tensor. All this is hard to reconcile with Einstein's claim to have derived the equations "in a completely formal manner."[79]

The claim, we suggest, should be understood against the backdrop of Einstein's obvious satisfaction that physical and mathematical considerations now seemed to point to the same field equations. Material in the Zurich Notebook shows that when Einstein began to generate field equations from their weak-field form by imposing energy-momentum conservation—the method that originally gave him the "Entwurf" field equations—he also tried to recover the resulting equations from the November tensor.[80] Such an alternative derivation of the field equations would have thrown light on their covariance properties. What Einstein presents in the review article of 1914 amounts to the same thing. Although he still had not found any connection between the "Entwurf" field equations and the Ricci tensor or the November tensor, he did supplement the physical considerations in the derivation of the "Entwurf" field equations by mathematical considerations that clarify—or so Einstein thought—their

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78 "Wir sind nun auf rein formalem Wege, d. h. ohne direkte Heranziehung unserer physikalischen Kenntnisse von der Gravitation, zu ganz bestimmten Feldgleichungen gelangt" (Einstein 1914c, 1076). Einstein had already announced this proudly in the introduction of the paper: "In particular, it was possible to obtain the equations for the gravitational field in a purely covariant-theoretical way" ("Es gelang insbesondere, die Gleichungen des Gravitationsfeldes auf einem rein kovarianten-theoretischen Wege zu gewinnen;" ibid., 1030). Earlier in 1914, in a paper co-authored with Adriaan D. Fokker, Einstein had made a similar claim for his reformulation of the Nordström theory in terms of Riemannian geometry. In the conclusion of their paper, the authors wrote: "In the foregoing it was possible to show that, if one bases oneself on the principle of the constancy of the velocity of light, one can arrive at Nordström's theory by purely formal considerations, i.e., without recourse to additional physical hypotheses" ("Im vorstehenden konnte gezeigt werden, daß man bei Zugrundelegung des Prinzips von der Konstanz der Lichtgeschwindigkeit durch rein formale Erwägungen, d.h. ohne Zuhilfenahme weiterer physikalischen Hypothesen zur Nordströmschen Theorie gelangen kann;" Einstein and Fokker 1914, 328).

79 Einstein likewise overestimated the importance of purely mathematical considerations in deriving the field equations of the Nordström theory in (Einstein and Fokker 1914).

80 See pp. 24R–25R of the Zurich Notebook and sec. 5.6 of "Commentary …" (this volume) for detailed analysis.

covariance properties. It therefore need not surprise us that Einstein overrated the importance of mathematical considerations in his new derivation of the "Entwurf" field equations.

5. A 'FATEFUL PREJUDICE' AND THE 'KEY TO THE SOLUTION': FROM THE "ENTWURF" LAGRANGIAN TO THE NOVEMBER LAGRANGIAN

When in the "Entwurf" Lagrangian the gravitational field is redefined as minus the Christoffel symbols, we find new field equations that bear a striking resemblance to field equations based on the November tensor found in the Zurich Notebook. When a factor $\sqrt{-g}$ *is omitted in the action with the new definition of the gravitational field, the field equations are exactly the same as these equations in the Zurich Notebook. Einstein called the old definition of the gravitational field a "fateful prejudice" and the new definition "the key to the solution." This strongly suggests that the way in which Einstein found his way back to these discarded field equations of the Zurich Notebook shortly before he published them in (Einstein 1915a) was essentially the same as the way in which they are recovered in this section.*

What are the field equations if one retains the form of (the gravitational part of) the "Entwurf" Lagrangian $Q = H\sqrt{-g}$, with

$$H = -g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} \tag{52}$$

(see eq. (36)), but changes the definition of the components of the gravitational field from

$$\Gamma^{\alpha}_{\beta\mu} \equiv \frac{1}{2}g^{\alpha\rho}g_{\rho\beta,\mu} \tag{53}$$

(see eq. (37)) to

$$\Gamma^{\alpha}_{\beta\mu} \equiv -\left\{\begin{matrix}\alpha \\ \beta\mu\end{matrix}\right\} = -\frac{1}{2}g^{\alpha\rho}(g_{\rho\beta,\mu} + g_{\rho\mu,\beta} - g_{\beta\mu,\rho}), \tag{54}$$

as Einstein did in his first November 1915 paper?

As before (see eqs. (36)-(45)), the left-hand side of the field equations follows from $\delta J = 0$, where

$$J = \int Q d\tau = \int H\sqrt{-g}d\tau. \tag{55}$$

Variation of Q gives two contributions (see eq. (38)):

$$\delta Q = \sqrt{-g}\delta H + (\delta\sqrt{-g})H. \tag{56}$$

The second contribution is the same as before (see eq. (40)):

$$(\delta\sqrt{-g})H = \left(\frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\right)\delta g^{\mu\nu}. \tag{57}$$

Likewise, δH can once again be written as

$$\delta H = -\delta g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} - 2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\delta\Gamma^{\beta}_{\alpha\nu}.$$

However, since $\Gamma^{\alpha}_{\beta\mu}$ in eq. (54) is symmetric in its lower indices whereas $\Gamma^{\alpha}_{\beta\mu}$ in eq. (53) is not, the expression for δH ends up being much simpler than before (cf. eq. (41)). The expression for δH above can be rewritten as

$$\delta H = \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\delta g^{\mu\nu} - 2\Gamma^{\alpha}_{\beta\mu}\delta(g^{\mu\nu}\Gamma^{\beta}_{\alpha\nu}),$$

which reduces to:[81]

$$\delta H = \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\delta g^{\mu\nu} - \Gamma^{\alpha}_{\mu\nu}\delta g^{\mu\nu}_{\alpha}. \tag{58}$$

Inserting eqs. (57) and (58) into eq. (56), one finds:

$$\delta Q = \sqrt{-g}\left[\left(\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\right)\delta g^{\mu\nu} - \Gamma^{\alpha}_{\mu\nu}\delta g^{\mu\nu}_{\alpha}\right]. \tag{59}$$

It follows that

$$\frac{\partial Q}{\partial g^{\mu\nu}} = \sqrt{-g}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}, \tag{60}$$

$$\frac{\partial Q}{\partial g^{\mu\nu}_{\alpha}} = -\sqrt{-g}\Gamma^{\alpha}_{\mu\nu}. \tag{61}$$

Inserting equations (60)–(61) into eq. (44), one finds the field equations:

$$\frac{\partial}{\partial x^{\alpha}}(\sqrt{-g}\Gamma^{\alpha}_{\mu\nu}) + \sqrt{-g}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}\sqrt{-g}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma} = \kappa T_{\mu\nu}. \tag{62}$$

If one omits the factors $\sqrt{-g}$ in the first two terms and uses eq. (54) for the components of the gravitational field, these two terms become:

$$-\frac{\partial}{\partial x^{\alpha}}\begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix} + \begin{Bmatrix}\alpha\\ \beta\mu\end{Bmatrix}\begin{Bmatrix}\beta\\ \nu\alpha\end{Bmatrix}.$$

This is just minus the November tensor which Einstein had extracted from the Ricci tensor in the Zurich Notebook by imposing the restriction to unimodular transformations. It is not hard to see how the calculation in eqs. (55)–(62) needs to be modified

81 Using the definition of $\Gamma^{\beta}_{\alpha\nu}$ in eq. (54), one can rewrite the last term of the expression above as

$$-\Gamma^{\alpha}_{\beta\mu}\delta(g^{\mu\nu}g^{\beta\lambda}(g_{\lambda\alpha,\nu} + g_{\lambda\nu,\alpha} - g_{\alpha\nu,\lambda}))$$

Since $\Gamma^{\alpha}_{\beta\mu}g^{\mu\nu}g^{\beta\lambda}$ is symmetric in λ and ν and $g_{\lambda\alpha,\nu} - g_{\nu\alpha,\lambda}$ is anti-symmetric in λ and ν, their contraction vanishes and the expression above reduces to:

$$-\Gamma^{\alpha}_{\beta\mu}\delta(g^{\mu\nu}g^{\beta\lambda}g_{\lambda\nu,\alpha}).$$

Using that $g^{\mu\nu}g^{\beta\lambda}g_{\lambda\nu,\alpha} = -g^{\mu\beta}_{\alpha}$, one can rewrite this as $\Gamma^{\alpha}_{\mu\nu}\delta g^{\mu\nu}_{\alpha}$.

in order to recover the field equations (6) based on the November tensor without any of the additional extra terms and factors in eq. (62). First, one only requires J to transform as a scalar under unimodular transformations whenever H does. One can then start from

$$J \equiv -\int H d\tau \,. \tag{63}$$

Before (see eq. (55)) a factor $\sqrt{-g}$ was needed because only the combination $\sqrt{-g}d\tau$ is an invariant volume element under arbitrary transformations. Under unimodular transformations, however, $d\tau$ by itself is invariant.

It turns out (see sec. 6) that omission of a factor $\sqrt{-g}$ in the action seriously complicates the use of the formalism of (Einstein 1914c). It would have been easier to work with the field equations (62) retaining all factors of $\sqrt{-g}$. The covariance properties of these equations, however, look as intractable as those of the "Entwurf" equations. Omission of a factor $\sqrt{-g}$ was a small price to pay for field equations with a broad well-defined covariance group closely connected to the Riemann tensor.[82] This was the connection Einstein had been looking for in vain in the days of the Zurich Notebook. In (Einstein 1914c) he thought that such a connection was no longer needed, that instead he could supplement the physical considerations going into the derivation of the "Entwurf" field equations with mathematical ones establishing, or so it seemed, that their covariance was broad enough for the generalization of the relativity principle he envisioned. That approach had ultimately failed. The action (63) now promised to resurrect the old ideal of the Zurich Notebook in which physical and mathematical considerations would point to the same field equations. Complications coming from omitting a factor $\sqrt{-g}$ could be dealt with later.

Eq. (63) is indeed the action from which the field equations are derived in (Einstein 1915a, 784, eq. 17). Einstein used the notation L for $-H$. L is given by (see eq. (52)):

$$\mathrm{L} = -H = g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} \,, \tag{64}$$

The field equations are:

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial \mathrm{L}}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial \mathrm{L}}{\partial g^{\mu\nu}} = -\kappa T_{\mu\nu} \tag{65}$$

82 Another advantage was that unimodular transformations are autonomous. Einstein had been greatly relieved when, in August 1913, he hit upon the (fallacious) argument that energy-momentum conservation limited the covariance of the "Entwurf" equations to linear transformations. That meant that he could stop searching for non-linear non-autonomous transformation leaving the equations invariant. He had not been unable to find a single one up to that point (Einstein to Lorentz, 14 August 1913 [CPAE 5, Docs. 467 and 470]; for further discussion, see sec. 3 of "What did Einstein know …" [this volume]). After the argument had evaporated, Einstein had been forced to reconsider non-autonomous transformations. He eventually concluded that the condition $B_{\mu} = 0$ for the "Entwurf" theory (see eq. (51)) allows non-autonomous transformations to arbitrarily moving systems. The simple case of rotation in Minkowski space-time proved him wrong. With field equations invariant under unimodular transformations, Einstein could avoid these problematic non-autonomous transformations altogether.

(Einstein 1915a, 784, eq. 18). The variation $\delta \mathrm{L} = -\delta H$ can be read off of eq. (58). It follows that (Einstein 1915a, 784, eqs. 19–19a):[83]

$$\frac{\partial \mathrm{L}}{\partial g^{\mu\nu}} = -\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}, \tag{66}$$

$$\frac{\partial \mathrm{L}}{\partial g^{\mu\nu}_{\alpha}} = \Gamma^{\alpha}_{\mu\nu}. \tag{67}$$

Inserting eqs. (66)–(67) into eq. (65), one finds the field equations (Einstein 1915a, 783, eq. 16a):

$$\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} = -\kappa T_{\mu\nu}. \tag{68}$$

When minus the Christoffel symbols are substituted for the quantities $\Gamma^{\alpha}_{\beta\mu}$, eq. (68) turns into eq. (6) based on the "November" tensor familiar from the Zurich Notebook. These field equations replace the "Entwurf" equations in (Einstein 1915a).

By changing the definition of the gravitational field in the "Entwurf" Lagrangian and by restricting the variational formalism of sec. 3 to unimodular transformations, we have thus arrived at field equations with a clearly defined covariance group. Einstein's comments that the old definition (53) of the gravitational field was a "fateful prejudice" (cf. note 38) and that the new definition (54) was "the key to the solution" (cf. note 39) provide strong textual evidence that he found his way back to the field equations (68) in essentially the same way in which they were derived in this section.

This provides a remarkably simple solution to one of the central puzzles in reconstructing Einstein's path to the Einstein field equations. As John Norton (1984, 142) put it: "Why Einstein should choose [the November tensor, i.e., the left-hand side of eq. (68)] as his gravitation tensor rather than a generally-covariant tensor, such as the Ricci tensor or even the Einstein tensor itself, has hitherto been a puzzle." Norton conjectured that it was Einstein's prejudice about the form of the metric for weak static fields that prevented him from choosing the Ricci tensor. To reduce the Ricci tensor to the d'Alembertian acting on the metric in the case of weak fields, the argu-

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83 Note that the operations 'doing the variations' and 'setting $\sqrt{-g} = 1$' do not commute. Setting $\sqrt{-g} = 1$ in eqs. (60)–(61)—i.e., *after* doing the variations in eqs. (55)–(59)—does not reduce these equations to eqs. (66)-(67), which are obtained by setting $\sqrt{-g} = 1$ in eq. (55)—i.e., *before* doing the variations. If the condition $\sqrt{-g} = 1$ is imposed first, the variation is done under a constraint. This is the analogue in functional analysis of the problem in ordinary calculus of finding the extrema of a function under some additional constraint. Such problems can be replaced by the problem of finding the extrema of a related function(al) without constraints through the well-known technique of Lagrange multipliers. Einstein was familiar with these techniques for functionals from his work in statistical mechanics. To derive various ensembles in statistical mechanics (micro-canonical, canonical, or grand canonical), one maximizes the entropy under two constraints, one on the total energy and one on the total particle number. The difference with the case of varying the action for the metric field is that the constraint $\sqrt{-g(x)} = 1$ has to be imposed at every point x, so that there is an infinite number of constraints. The Lagrange multipliers thus become a new field. Techniques for doing this have been worked out in the context of what has come to be known as unimodular gravity, a theory first proposed in (Einstein 1919) and first cast in Lagrangian form in (Anderson and Finkelstein 1971).

ment went, one needs the harmonic coordinate condition, which is not satisfied by Einstein's metric for weak static fields. In the case of the November tensor, one can use the Hertz condition for this purpose (see eqs. (7)-(8)), which is satisfied by Einstein's metric for weak static fields. Aside from solving Norton's puzzle, there is no evidence that the incompatibility between the harmonic condition and Einstein's prejudice about the form of the metric for weak static fields played a role at this juncture. Our alternative solution to the puzzle removes the need for invoking this incompatibility.[84] Why did Einstein choose the November tensor rather than the Ricci tensor? Because both the mathematical and the physical strategy he employed in his search for suitable field equations pointed to the November tensor, not to the Ricci tensor.

6. THE FIRST NOVEMBER PAPER: THE KNOT UNTIED

In the Zurich Notebook Einstein had not been able to show that field equations based on the November tensor are compatible with energy-momentum conservation. In 1915 the variational formalism of (Einstein 1914c) showed him how to solve this problem. He essentially just had to find the conditions $B_\mu = 0$ *for this specific Lagrangian. As we saw in sec. 3, such conditions also determine the covariance properties of the field equations. Because of the way in which the November tensor can be extracted from the Ricci tensor, it is clear that the field equations based on the November tensor are invariant under unimodular transformations. One would therefore expect that in this case the four conditions* $B_\mu = 0$ *reduce to one condition expressing the restriction to unimodular transformations. The four conditions can indeed be replaced by one, but this one condition says that* $\sqrt{-g}$ *can not be a constant. This is more restrictive than the condition that* $\sqrt{-g}$ *transform as a scalar. It was nonetheless an important result that the compatibility of the field equations with energy-momentum conservation only called for one additional condition. It still takes four conditions to show that the field equations reduce to the Poisson equation for weak static fields. We suggest that this made it clear to Einstein that he could use coordinate conditions in the modern sense to recover the Poisson equation and that a coordinate restriction was needed only for energy-momentum conservation.*

According to the general formalism of (Einstein 1914c), the gravitational field equations are compatible with energy-momentum conservation if the conditions $S_\sigma^\nu = 0$ and $B_\mu = 0$ are satisfied (see eqs. (20)–(24)). These conditions, however, were derived for an action of the form $\int\sqrt{-g}H d\tau$. The field equations of (Einstein 1915a) were derived from an action of the form $\int \mathrm{L} d\tau$ without the factor $\sqrt{-g}$ (see eqs. (63)–(64)). This seriously complicates matters (see notes 83 and 88) and in his papers of November 1915, as Einstein realized, he could not simply apply the formalism. He nonetheless relied heavily on the formalism to guide him in his analysis of

84 Norton (1984, 102) invoked this same incompatibility to explain why Einstein abandoned field equations based on the Ricci tensor in the Zurich Notebook. As we mentioned earlier (see note 10), we see no evidence that it played any role there either.

the new theory. In (Einstein 1915a, 784–785), for instance, he went through a calculation closely analogous to the one in the general formalism (see sec. 3.2, eqs. (17)–(23)) to establish that the field equations derived from $\int \mathrm{L} d\tau$ are compatible with energy-momentum conservation under the restriction to unimodular transformations.

Einstein (1915a, sec. 1–2) first exploited the restriction to unimodular transformations to replace the energy-momentum balance equation, $T^{\lambda}_{\sigma;\lambda} = 0$, by a simpler equation. The covariant divergence of $T_{\mu\nu}$, the energy-momentum tensor of matter, is given by (see note 35):

$$T^{\lambda}_{\sigma;\lambda} = T^{\lambda}_{\sigma,\lambda} + \begin{Bmatrix} \lambda \\ \mu\lambda \end{Bmatrix} T^{\mu}_{\sigma} - \begin{Bmatrix} \mu \\ \sigma\lambda \end{Bmatrix} T^{\lambda}_{\mu}. \tag{69}$$

The second term on the right-hand side can be rewritten as (see note 35):

$$\begin{Bmatrix} \lambda \\ \mu\lambda \end{Bmatrix} T^{\mu}_{\sigma} = (\log\sqrt{-g})_{,\mu} T^{\mu}_{\sigma};$$

the third term as (see note 36):

$$\begin{Bmatrix} \mu \\ \sigma\lambda \end{Bmatrix} T^{\lambda}_{\mu} = \frac{1}{2} g^{\mu\nu} g_{\nu\lambda,\sigma} T^{\lambda}_{\mu} = -\frac{1}{2} g^{\mu\nu}_{\sigma} T_{\mu\nu}.$$

Inserting both expressions into eq. (69) and regrouping terms, one finds

$$T^{\lambda}_{\sigma;\lambda} = \left[T^{\lambda}_{\sigma,\lambda} + \frac{1}{2} g^{\mu\nu}_{\sigma} T_{\mu\nu} \right] + (\log\sqrt{-g})_{,\mu} T^{\mu}_{\sigma}. \tag{70}$$

The left-hand side of eq. (70) is a generally-covariant vector. The last term on the right-hand side transforms as a vector under unimodular transformations. It follows that the expression in square brackets must also transform as a vector under unimodular transformations. In (Einstein 1915a) the vanishing of this expression is used as the energy-momentum balance law:

$$T^{\lambda}_{\sigma,\lambda} = -\frac{1}{2} g^{\mu\nu}_{\sigma} T_{\mu\nu}. \tag{71}$$

Note that eq. (71) is *not* equivalent to $T^{\lambda}_{\sigma;\lambda} = 0$, unless the last term of eq. (70) vanishes, as it does when the restriction to unimodular transformations is strengthened by setting $\sqrt{-g} = 1$.[85]

Einstein (1915a, 784–785) investigated whether any restrictions over and above the restriction to unimodular transformations would be needed to make sure that the field equations (68) are compatible with energy-momentum conservation as expressed in eq. (71). Using the field equations (65) on the right-hand side of eq. (71), one finds (cf. eq. (18)):

$$T^{\lambda}_{\sigma,\lambda} = \frac{1}{2\kappa} g^{\mu\nu}_{\sigma} \left(\frac{\partial}{\partial x^{\alpha}} \left(\frac{\partial \mathrm{L}}{\partial g^{\mu\nu}_{\alpha}} \right) - \frac{\partial \mathrm{L}}{\partial g^{\mu\nu}} \right). \tag{72}$$

This can be rewritten as (Einstein 1915a, eq. 20)

$$(T_\sigma^\lambda + t_\sigma^\lambda)_{,\lambda} = 0\,,$$

by introducing a gravitational energy-momentum pseudo-tensor defined as (ibid., eq. 20a):[86]

$$t_\sigma^\lambda = \frac{1}{2\kappa}\left(\delta_\sigma^\lambda \mathrm{L} - g_\sigma^{\mu\nu}\frac{\partial \mathrm{L}}{\partial g_\lambda^{\mu\nu}}\right). \tag{73}$$

Inserting eq. (64) for L and eq. (67) for its derivative with respect to $g_\lambda^{\mu\nu}$, one finds

$$\kappa t_\sigma^\lambda = \frac{1}{2}(\delta_\sigma^\lambda[g^{\mu\nu}\Gamma_{\beta\mu}^\alpha\Gamma_{\alpha\nu}^\beta] - g_\sigma^{\mu\nu}\Gamma_{\mu\nu}^\lambda)\,,$$

which can be rewritten as (ibid, eq. 20b)[87]

$$\kappa t_\sigma^\lambda = \frac{1}{2}\delta_\sigma^\lambda g^{\mu\nu}\Gamma_{\beta\mu}^\alpha\Gamma_{\alpha\nu}^\beta - g^{\mu\nu}\Gamma_{\mu\sigma}^\alpha\Gamma_{\alpha\nu}^\lambda. \tag{74}$$

Einstein now rewrote (the mixed form of) the field equations in such a way that they have $T_\mu^\lambda + t_\mu^\lambda$ on the right-hand side. The divergence of the left-hand side then gives the quantity B_μ in the new theory. The vanishing of B_μ in conjunction with the field equations guarantees energy-momentum conservation, i.e., the vanishing of the ordinary divergence of $T_\mu^\lambda + t_\mu^\lambda$.[88]

Contraction of the field equations (68) with $g^{\nu\lambda}$ gives:

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85 Einstein explicitly acknowledged that eq. (71) is not equivalent to $T_{\sigma;\lambda}^\lambda = 0$. The restriction to unimodular transformations, Einstein (1915a, 780) conceded, cannot be used to simplify the basic formulae for covariant differentiation given in his systematic exposition of the "Entwurf" theory (Einstein 1914c, 1050, eqs. (29) and (30)). But, he added, the "defining definition" ("Definitionsgleichung") of the covariant divergence can be simplified. He then went through the argument following eq. (70) to redefine the covariant divergence of an arbitrary symmetric tensor $A^{\mu\nu}$ (Einstein 1915a, 780, eq. 9) as: $\partial A_\sigma^\nu/\partial x^\nu - (1/2)g^{\tau\mu}g_{\mu\nu,\sigma}A_\tau^\nu$. He noted (ibid., 781) that this equation has the same form as the covariant divergence of the tensor density $\mathrm{A}^{\mu\nu} \equiv \sqrt{-g}A^{\mu\nu}$ as defined in (Einstein 1914c, 1054, eq. 41b). This illustrates the general remark he made at the beginning of sec. 1 of (Einstein 1915a): "Because of the scalar character of $\sqrt{-g}$ a simplification of the basic formulae for the formation of invariant objects is possible … which in short comes down to this, that the factors $\sqrt{-g}$ and $1/\sqrt{-g}$ no longer occur in the basic formulae and that the difference between tensors and tensor densities disappears" ("Vermöge des Skalarscharakters von $\sqrt{-g}$ lassen die Grundformeln der Kovariantenbildung … eine Vereinfachung zu, die kurz gesagt darin beruht, daß in den Grundformeln die Faktoren $\sqrt{-g}$ und $1/\sqrt{-g}$ nicht mehr auftreten und der Unterschied zwischen Tensoren und V-Tensoren wegfällt.").

86 The derivation of eq. (73) is fully analogous to the derivation of eq. (19) in the general formalism (replace Q by L in the calculation in note 32).

87 Since the covariant derivative of the metric vanishes,

$$0 = g^{\mu\nu}{}_{;\sigma} = g_\sigma^{\mu\nu} + \left\{ {\mu \atop \alpha\sigma} \right\} g^{\alpha\nu} + \left\{ {\nu \atop \alpha\sigma} \right\} g^{\mu\alpha} = g_\sigma^{\mu\nu} - \Gamma_{\alpha\sigma}^\mu g^{\alpha\nu} - \Gamma_{\alpha\sigma}^\nu g^{\mu\alpha},$$

it follows that $g_\sigma^{\mu\nu}\Gamma_{\mu\nu}^\lambda = (\Gamma_{\alpha\sigma}^\mu g^{\alpha\nu} + \Gamma_{\alpha\sigma}^\nu g^{\mu\alpha})\Gamma_{\mu\nu}^\lambda = 2g^{\alpha\nu}\Gamma_{\alpha\sigma}^\mu\Gamma_{\mu\nu}^\lambda = 2g^{\mu\nu}\Gamma_{\mu\sigma}^\alpha\Gamma_{\alpha\nu}^\lambda$.

$$g^{\nu\lambda}(\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}) = -\kappa T^{\lambda}_{\mu}\,.$$

This can be rewritten as:[89]

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\nu\rho}\Gamma^{\lambda}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} = -\kappa T^{\lambda}_{\mu}\,.$$

The second term on the left-hand side is equal to the last term of eq. (74). Hence:

$$-g^{\nu\rho}\Gamma^{\lambda}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} = \kappa t^{\lambda}_{\mu} - \frac{1}{2}\delta^{\lambda}_{\mu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\,.$$

The field equations can thus be written as:[90]

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - \frac{1}{2}\delta^{\lambda}_{\mu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma} = -\kappa(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})\,. \tag{75}$$

The quantity B_μ in this case is thus given by (cf. eqs. (22)–(23)):

$$B_\mu = \frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial}{\partial x^{\alpha}}(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu}) - \frac{1}{2}\delta^{\lambda}_{\mu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}\right)\,. \tag{76}$$

The condition $B_\mu = 0$ guarantees that $(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})_{,\lambda} = 0$.

Given his analysis of the covariance properties of the “Entwurf” field equations in 1914, Einstein had come to expect that this same condition $B_\mu = 0$ circumscribes the covariance of the field equations. In the case of field equations (68), he knew that their covariance group is that of arbitrary unimodular transformations, i.e., transformations under which the determinant g of the metric transforms as a scalar. This

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88 Einstein could not just replace Q by $H = -\mathrm{L}$ in the field equations in the form of eq. (13) and read off B_μ from the resulting equations. Recall that there are two definitions of the gravitational energy-momentum tensor, designated earlier as $t^{\lambda}_{\mu}(Q, \text{source})$ and $t^{\lambda}_{\mu}(Q, \text{cons})$ (cf. eqs. (20)–(21)). The condition $S^{\nu}_{\sigma}(Q) = 0$ guarantees that these two quantities are equal to one another. The corresponding quantities $t^{\lambda}_{\mu}(\mathrm{L}, \text{source})$ and $t^{\lambda}_{\mu}(\mathrm{L}, \text{cons})$, however, are not equal to one another, since $S^{\nu}_{\sigma}(\mathrm{L}) \neq 0$ (cf. the discussion following eq. (35)).The analogue of eq. (13) in this case would be:

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\nu\lambda}\frac{\partial\mathrm{L}}{\partial g^{\mu\nu}_{\alpha}}\right) = -\kappa(T^{\lambda}_{\mu} + t^{\lambda}_{\mu}(\mathrm{L}, \text{source}))$$

The gravitational energy-momentum pseudo-tensor in eq. (74) corresponds to $t^{\lambda}_{\mu}(\mathrm{L}, \text{cons})$ (as will be clear from comparing eq. (73) to eq. (19)). This quantity cannot be used interchangeable with $t^{\lambda}_{\mu}(\mathrm{L}, \text{source})$.

89 The left-hand side can be written as

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\nu\lambda}_{\alpha}\Gamma^{\alpha}_{\mu\nu} + g^{\nu\lambda}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\,.$$

Substituting $\Gamma^{\nu}_{\alpha\rho}g^{\rho\lambda} + \Gamma^{\lambda}_{\alpha\rho}g^{\nu\rho}$ for $g^{\nu\lambda}_{\alpha}$ in the second term (see note 87), one finds:

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\rho\lambda}\Gamma^{\nu}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} - g^{\nu\rho}\Gamma^{\lambda}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} + g^{\nu\lambda}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$$

The second term cancels against the fourth and the two remaining terms form the left-hand side of the equation below.

90 Einstein omitted the manipulations to get to eq. (75). He simply wrote: “after simple rearrangement” (“nach einfacher Umformung,” Einstein 1915a, 785). The second term on the left-hand side of eq. (75) is, as we shall see later (see eq. (86)), equal to $-(1/2)\delta^{\lambda}_{\mu}t$, where t is the trace of t^{λ}_{μ}.

only gives one condition, not four as in eq. (76). In view of this mismatch, it is understandable that Einstein tried to replace these four conditions by one condition on g.[91]

His first step was to rewrite B_μ in the form $\partial A / \partial x_\mu$ and make B_μ vanish by imposing the stronger condition $A = 0$. Eq. (76) can be rewritten as

$$B_\mu = \frac{\partial^2}{\partial x^\lambda \partial x^\alpha}(g^{\nu\lambda}\Gamma^\alpha_{\mu\nu}) - \frac{1}{2}\frac{\partial}{\partial x^\mu}(g^{\rho\sigma}\Gamma^\alpha_{\beta\rho}\Gamma^\beta_{\alpha\sigma}) .$$

The first term works out to be $\frac{1}{2}g^{\alpha\beta}_{\alpha\beta\mu}$,[92] so this equation becomes (Einstein 1915a, 785, eq. 22[93])

$$B_\mu = \frac{1}{2}\frac{\partial}{\partial x^\mu}[g^{\alpha\beta}_{\alpha\beta} - g^{\rho\sigma}\Gamma^\alpha_{\beta\rho}\Gamma^\beta_{\alpha\sigma}] . \tag{77}$$

Einstein now replaced the four conditions $B_\mu = 0$ by a single stronger condition (ibid., eq. 22a):

$$g^{\alpha\beta}_{\alpha\beta} - g^{\rho\sigma}\Gamma^\alpha_{\beta\rho}\Gamma^\beta_{\alpha\sigma} = 0 . \tag{78}$$

The next step was to replace this condition by a condition on g. To this end, Einstein fully contracted the field equations and compared the resulting condition to condition (78). Contraction of the field equations (68) with $g^{\mu\nu}$ gives:

$$g^{\mu\nu}(\Gamma^\alpha_{\mu\nu,\alpha} + \Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu}) = -\kappa T . \tag{79}$$

This equation can be rewritten as

$$(g^{\mu\nu}\Gamma^\alpha_{\mu\nu})_{,\alpha} - g^{\mu\nu}_\alpha\Gamma^\alpha_{\mu\nu} + g^{\mu\nu}\Gamma^\alpha_{\beta\mu}\Gamma^\beta_{\alpha\nu} = -\kappa T . \tag{80}$$

The first term on the left-hand side is equal to:[94]

$$g^{\alpha\beta}_{\alpha\beta} + \frac{\partial}{\partial x^\alpha}\left(g^{\alpha\beta}\frac{\partial}{\partial x^\beta}\log\sqrt{-g}\right) ;$$

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91 As we see no other plausible explanation for this move, this provides strong evidence for our claim that Einstein relied heavily on the formalism of (Einstein 1914c) to guide him in the analysis presented in the papers of November 1915.

92 Inserting eq. (54) for $\Gamma^\alpha_{\mu\nu}$, one finds

$$\frac{\partial^2}{\partial x^\lambda \partial x^\alpha}(g^{\nu\lambda}\Gamma^\alpha_{\mu\nu}) = -\frac{1}{2}\frac{\partial^2}{\partial x^\lambda \partial x^\alpha}(g^{\nu\lambda}g^{\alpha\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho}))$$

Using that $g^{\mu\alpha}g_{\alpha\nu,\lambda} = -g^{\mu\alpha}_\lambda g_{\alpha\nu}$, one can rewrite this as:

$$-\frac{1}{2}\frac{\partial^2}{\partial x^\lambda \partial x^\alpha}(-g^{\nu\lambda}g^{\alpha\rho}_\nu g_{\rho\mu} - g^{\nu\lambda}g^{\alpha\rho}_\mu g_{\rho\nu} + g^{\nu\lambda}_\rho g^{\alpha\rho}g_{\mu\nu}) .$$

The first and the third term in parentheses can be grouped together to form a quantity anti-symmetric in λ and α and thus vanish upon contraction with $\partial^2 / \partial x^\lambda \partial x^\alpha$; the second term can be rewritten as $\delta^\lambda_\rho g^{\alpha\rho}_\mu = g^{\alpha\lambda}_\mu$. Finally, $(g^{\alpha\lambda}_\mu)_{,\lambda\alpha} = g^{\alpha\lambda}_{\alpha\lambda\mu}$.

93 Einstein did not use the designation B_μ for this quantity, thereby obscuring its origin in the variational formalism of (Einstein 1914c).

the second to minus twice the third:[95]

$$-2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\,.$$

Inserting these last two expressions into eq. (80), one finds (Einstein 1915a, 785, eq. 21[96]):

$$g^{\alpha\beta}_{\alpha\beta} - g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\log\sqrt{-g}\right) = -\kappa T\,. \tag{81}$$

The first two terms on the left-hand side of this equation are equal to the two terms on the left-hand side of eq. (78). Given that eq. (81) follows from the field equations (68), it suffices to demand that

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\log\sqrt{-g}\right) = -\kappa T \tag{82}$$

(ibid., 785, eq. 21a) to make sure that condition (78) is also satisfied. Through eq. (77) this guarantees that $B_{\mu} = 0$. This in turn guarantees that $(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})_{,\lambda} = 0$ (see eqs. (75)–(76)). One condition on the determinant of the metric thus suffices to guarantee compatibility of the field equations (68) with energy-momentum conservation. The condition is an odd one though. The energy-momentum tensor phenomenologically representing ordinary matter has a non-vanishing trace. Eq. (82) thus says that g cannot be a constant. This means that there still is a residual discrepancy between the covariance of the field equations and the coordinate restriction needed to guarantee compatibility with energy-momentum conservation. The restriction to unimodular transformations only demands g to transform as a scalar, it does not say that it cannot be a constant. Within a few weeks, Einstein published two modifications of his field equations to change this condition to the more congenial condition $\sqrt{-g} = 1$ for unimodular coordinates (see sec. 7).

It was nonetheless an extremely important result that the four conditions $B_{\mu} = 0$ can be reduced to one condition in this case. Up to this point, Einstein had not made a distinction between coordinate restrictions necessitated by the demand that the field equations reduce to the Poisson equation for weak static fields and restrictions necessitated by the demand that they be compatible with energy-momentum conservation.

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94 Inserting eq. (54) for $\Gamma^{\alpha}_{\mu\nu}$, one finds (cf. note 92):

$$(g^{\mu\nu}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} = -\frac{1}{2}\frac{\partial}{\partial x^{\alpha}}(g^{\mu\nu}g^{\alpha\rho}(2g_{\rho\mu,\nu} - g_{\mu\nu,\rho}))\,.$$

The first term on the right-hand side can be rewritten as $(g^{\mu\nu}g^{\alpha\rho}_{\nu}g_{\rho\mu})_{,\alpha} = g^{\alpha\nu}_{\nu\alpha}$; the second term as

$$\frac{1}{2}\frac{\partial}{\partial x^{\alpha}}(g^{\alpha\rho}[g^{\mu\nu}g_{\mu\nu,\rho}]) = \frac{1}{2}\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\rho}\left[2\frac{\partial}{\partial x^{\rho}}\log\sqrt{-g}\right]\right)$$

(see, e.g., Einstein 1914c, 1051, eq. 32). The sum of these two terms gives the expression below.

95 Substituting $\Gamma^{\mu}_{\alpha\rho}g^{\rho\nu} + \Gamma^{\nu}_{\alpha\rho}g^{\mu\rho}$ for $g^{\mu\nu}_{\alpha}$ in $-g^{\mu\nu}_{\alpha}\Gamma^{\alpha}_{\mu\nu}$ (cf. note 87), one finds $-2g^{\rho\nu}\Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu}$. Relabeling of the summation indices gives the expression below.

96 Einstein omitted the manipulations to get from eq. (79) to eq. (81), again writing simply “after simple rearrangement” (cf. note 90)

It now turned out that the latter demand could be satisfied by one condition whereas the former continued to call for four. We conjecture that this drove home the point that the two requirements should be dealt with separately.

In fact, immediately after eq. (82) in (Einstein 1915a), we find the very first unambiguous instance in both Einstein's published papers and extant manuscripts and correspondence of a coordinate condition used in the modern sense. Einstein (1915a, 786, eqs. 22 and 16b) showed how the conditions $g^{\alpha\beta}_{\beta} = 0$, which we have called the Hertz condition/restriction (see eq. (7)), reduce the November tensor to the d'Alembertian acting on the metric in the case of weak fields. Einstein had done the same calculation in the Zurich Notebook (see eqs. (7)–(8)). There he had used $g^{\alpha\beta}_{\beta} = 0$ as a coordinate restriction. As such it was unacceptable because it was not satisfied, for instance, by the Minkowski metric in rotating coordinates.[97] The return of $g^{\alpha\beta}_{\beta} = 0$ in (Einstein 1915a) makes it clear that Einstein did not see this as a problem anymore in 1915. Einstein had come to realize that the conditions $g^{\alpha\beta}_{\beta} = 0$ are not an integral part of the theory and only serve to facilitate comparison of the field equations to the Poisson equation of Newtonian theory in the case of weak static fields. In other words, Einstein now saw that $g^{\alpha\beta}_{\beta} = 0$ is not a coordinate restriction but a coordinate condition in the modern sense.

The theory of (Einstein 1915a) was thus of broad covariance. The only restrictions were that the determinant of the metric transform as a scalar and that it not be a constant. The way Einstein saw it, this was all that was needed to solve the problem of rotation that had brought down the "Entwurf" theory.[98] In the concluding paragraphs of (Einstein 1915a), he pointed out that transformations to rotating coordinates belong to the class of unimodular coordinates under which the new field equations are invariant.[99]

Looking back on secs. 5 and 6, we can clearly see how redefining the components of the gravitational field untied the tight knot of conditions and definitions that had been the "Entwurf" theory and retied it in a slightly different way to become a theory within hailing distance of general relativity as we know it today. First and foremost, the redefinition of the gravitational field led to the replacement of the "Entwurf" field equations and their intractable covariance properties by field equations invariant

97 John Norton (1984, 119 and 143) already suggested that Einstein rejected the combination of the November tensor and the conditions $g^{\alpha\beta}_{\beta} = 0$ in the Zurich Notebook because $g^{\alpha\beta}_{\beta} \neq 0$ for the Minkowski metric in rotating coordinates. We agree. Note, however, that Einstein's argument is cogent only if the conditions $g^{\alpha\beta}_{\beta} = 0$ are seen as a coordinate restriction rather than a coordinate condition. In fact, it was in an attempt to make sense of these remarks in (Norton 1984) that one of us (JR) first hit upon the distinction between coordinate conditions and what we have come to call coordinate restrictions.

98 In fact, the problem of rotation persisted even in the final version of general relativity. General covariance does not make rotation—or *any* non-geodesic motion for that matter—relative (see note 1).

99 On the face of it, it may seem that the theory still does not allow the Minkowski metric in rotating coordinates because its determinant equals one. Since the Minkowski metric is a vacuum solution of the field equations, however, it does not matter in this case that through eq. (82) $\sqrt{-g} = 1$ implies that $T = 0$ as well.

under arbitrary unimodular transformations. But that was not all. In the new theory, instead of the four additional restrictions $B_\mu = 0$ familiar from the "Entwurf" theory, it took only a minimal strengthening of the restriction to unimodular transformations (namely that the determinant of the metric not be a constant) to ensure that these new field equations yield energy-momentum conservation. Finally, because energy-momentum conservation only called for one extra condition whereas recovery of the Poisson equation continued to call for four, it became clear that these two types of conditions have a different status. Taking advantage of this insight, Einstein used a coordinate condition in the modern sense to show that the new field equations reduce to the Poisson equation for weak static fields and only used a coordinate restriction to satisfy the demands of energy-momentum conservation. There was thus enough covariance left in the new theory to meet the demands of Einstein's relativity and equivalence principles.

7. FROM THE NOVEMBER TENSOR TO THE EINSTEIN TENSOR

Einstein soon found ways to replace the condition of (Einstein 1915a) that $\sqrt{-g}$ cannot be a constant by the more congenial condition $\sqrt{-g} = 1$ for unimodular coordinates. In (Einstein 1915b), he achieved this through the assumption that the trace T of the energy-momentum tensor of matter vanishes. He justified this assumption by adopting the view, promoted by Gustav Mie and others, that all matter is electromagnetic. Energy-momentum conservation now followed from the field equations in unimodular coordinates without any additional coordinate restrictions. And the field equations themselves could be looked upon as generally-covariant field equations in unimodular coordinates. So Einstein had found generally-covariant field equations at last: $R_{\mu\nu} = -\kappa T_{\mu\nu}$ (with $R_{\mu\nu}$ the Ricci tensor). In his calculations, both in November 1915 and in (Einstein 1916a), however, he continued to use unimodular coordinates. And he soon had second thoughts about paying for general covariance by committing himself to the electromagnetic view of nature. In (Einstein 1915d), he changed the condition that $\sqrt{-g}$ cannot be a constant to the condition $\sqrt{-g} = 1$ without imposing any restrictions on T. This he achieved by adding a term with T to the right-hand side of the field equations based on the November tensor. He realized that this trace term was needed anyway to ensure that the energy-momentum tensor for matter enters the field equations in the exact same way as the gravitational energy-momentum pseudo-tensor. This told Einstein that these were the equations he had been looking for. As before, they could be looked upon as generally-covariant equations expressed in unimodular coordinates. Einstein had thus found the Einstein field equations: $R_{\mu\nu} = -\kappa(T_{\mu\nu} - (1/2)g_{\mu\nu}T)$.

In the second and fourth of his communications to the Berlin Academy in November 1915, Einstein (1915b, 1915d) proposed two different ways to avoid the requirement that $\sqrt{-g}$ cannot be a constant found in the first communication (Einstein 1915a). In the second November paper, a three-page "Addendum" ("Nachtrag") to the first, he assumed that all matter is electromagnetic, in which case $T = 0$ (Einstein 1915b). Condition (82) can then be satisfied by setting $\sqrt{-g} = 1$, the condition for unimodular coordinates. This has three advantages. First, by looking upon the field equations as holding only in unimodular coordinates (rather than in coordinates related to one another by unimodular transformations) he removed the residual discrepancy between the covariance of the field equations and the restriction needed to guarantee energy-momentum conservation. Second, with $\sqrt{-g} = 1$, the energy-momentum balance equation, $T^{\mu}_{\nu;\mu} = 0$, reduces to $T^{\mu}_{\nu,\mu} + (1/2)g^{\alpha\beta}_{\nu}T_{\alpha\beta} = 0$ (see eq. (71)), the equations used in the analysis of energy-momentum conservation in (Einstein 1915a). The third and most important advantage is that the field equations (68) could now be looked upon as generally-covariant field equations expressed in unimodular coordinates. Setting $\sqrt{-g} = 1$ also has one disadvantage. It rules out a metric of the form $g_{\mu\nu} = \mathrm{diag}(-1, -1, -1, f(x, y, z))$, which Einstein still assumed was the general form of the metric for weak static fields. Either Einstein did not think of this disadvantage at this point (though we shall see that he had thought of it a week later), or it was outweighed, at least for the time being, by the advantages.

Einstein (1915b, 800, eq. 16b) wrote these generally-covariant field equations as

$$G_{\mu\nu} = -\kappa T_{\mu\nu}, \tag{83}$$

where $G_{\mu\nu}$ is the Ricci tensor. As in (Einstein 1915a), he wrote the Ricci tensor as the sum of two terms,[100]

$$G_{im} = R_{im} + S_{im}.$$

The first term is defined as minus what we called the November tensor (i.e., T^{x}_{il} in eq. (5)):

$$R_{im} \equiv -\frac{\partial}{\partial x^l}\begin{Bmatrix} l \\ im \end{Bmatrix} + \begin{Bmatrix} \rho \\ il \end{Bmatrix}\begin{Bmatrix} l \\ \rho m \end{Bmatrix}.$$

The second term is defined as:

$$S_{im} \equiv \frac{\partial}{\partial x^m}\begin{Bmatrix} l \\ il \end{Bmatrix} - \begin{Bmatrix} \rho \\ im \end{Bmatrix}\begin{Bmatrix} l \\ \rho l \end{Bmatrix}.$$

Since the first and the third Christoffel symbol in this expression are equal to the gradient of $\lg\sqrt{-g}$ it follows that $S_{im} = 0$ in unimodular coordinates. In the Zurich Notebook and in his first November paper, Einstein used the decomposition of the Ricci tensor only to show that the November tensor transforms as a tensor under unimodular *transformations*. In unimodular *coordinates*, the Ricci tensor actually reduces to the November tensor and the field equations (83) reduce to the field equations (68) of (Einstein 1915a):

$$R_{\mu\nu} = -\kappa T_{\mu\nu} \tag{84}$$

100 See Einstein 1915a, 782; 1915b, 800; 1915d, 844.

(Einstein 1915b, 801, eq. 16).

Einstein had thus finally found generally-covariant field equations. His calculations in (Einstein 1915a) show that *in unimodular coordinates* these field equations guarantee energy-momentum conservation (see eqs. (71)–(86)). Although Einstein did not explicitly show this, it was reasonable to assume that the corresponding generally-covariant equations $G_{\mu\nu} = -\kappa T_{\mu\nu}$ guarantee energy-momentum conservation in arbitrary coordinates.

Most of the "Addendum" (Einstein 1915b) is taken up by a defense of the assumption $T = 0$. The results reported in the "Addendum" all depend on this assumption. The assumption holds for electromagnetic fields in Maxwell's theory and might continue to hold in the non-linear generalizations of Maxwell's theory pursued by Gustav Mie and other proponents of the electromagnetic view of nature.[101] It does not hold for the energy-momentum tensor routinely used to give a phenomenological description of ordinary matter. To get around this problem, Einstein assumed that gravitational fields play an important role in the constitution of matter. In that case the energy-momentum tensor phenomenologically describing matter should not be identified with $T^{\mu\nu}$ but with $T^{\mu\nu} + t^{\mu\nu}$ and the non-vanishing trace might come from $t^{\mu\nu}$ rather than $T^{\mu\nu}$. Einstein's flirtation with the electromagnetic world picture was short-lived, but three and a half years later he resurrected the idea that gravity plays a role in the structure of matter in a theory that makes the cosmological constant of (Einstein 1917) responsible both for the stability of the cosmos and the stability of elementary particles (Einstein 1919).[102]

At first Einstein was very enthusiastic about the electromagnetic turn his theory had taken. In an abstract for his third paper of November 1915 (Einstein 1915c)—the one in which he used the field equations of (Einstein 1915b) in unimodular coordinates to explain the anomalous advance of the perihelion of Mercury—Einstein wrote that this result "confirms the hypothesis of the vanishing of the scalar of the energy tensor of "matter" [i.e., $T = 0$]."[103] His enthusiasm, however, waned quickly. In a footnote to the perihelion paper itself, he announced:

> In a communication that will follow shortly it will be shown that this hypothesis [i.e., $T = 0$] is dispensable. Essential is only that a choice of reference frame is possible such that the determinant $|g_{\mu\nu}|$ takes on the value -1.[104]

As we shall see, the calculation of the perihelion advance of Mercury played an

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101 Einstein knew about Hilbert's work along these lines (see Renn and Stachel 1999 and Sauer 1999).

102 This theory is enjoying renewed interest. It now goes by the name of "unimodular gravity" (see Anderson and Finkelstein 1971; for a more recent discussion and further references, see Finkelstein et al. 2002, Earman 2003).

103 "Dadurch wird die Hypothese vom Verschwinden des Skalars des Energietensors der „Materie" bestätigt." *Königlich Preußische Akademie der Wissenschaften* (Berlin). *Sitzungsberichte* (1915): 803. For further discussion, see (Renn and Stachel 1999) and "Pathways …" (this volume).

104 "In einer bald folgenden Mitteilung wird gezeigt werden, daß jene Hypothese entbehrlich ist. Wesentlich ist nur, daß eine solche Wahl des Bezugssystems möglich ist, daß die Determinante $|g_{\mu\nu}|$ den Wert -1 annimmt" (Einstein 1915c, 831, note 1).

important role in showing Einstein that one can set $\sqrt{-g} = 1$ without setting $T = 0$.

In (Einstein 1915d), the fourth and final paper of November 1915 and the communication announced in the footnote quoted above, Einstein changed the field equations in such a way that condition (82) changes to

$$\frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\log\sqrt{-g}\right) = 0\,. \tag{85}$$

This makes it possible to choose unimodular coordinates (i.e., set $\sqrt{-g} = 1$) without putting any condition on the trace T of the energy-momentum tensor.

How did Einstein arrive at this new condition (85)? Recall that the original condition (82) followed from combining two equations. The first is eq. (78) which comes from the condition $B_{\mu} = 0$ for the field equations (68) of (Einstein 1915a). The second is eq. (81) which comes from fully contracting those field equations. These two equations can be written more compactly by introducing the trace of the gravitational energy-momentum pseudo-tensor (74) as Einstein first did in (Einstein 1915d, 846):[105]

$$\kappa t = g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}\,. \tag{86}$$

Inserting κt in the second terms of both eq. (78) and eq. (81), one finds:

$$g^{\alpha\beta}_{\alpha\beta} - \kappa t = 0\,, \tag{87}$$

$$g^{\alpha\beta}_{\alpha\beta} - \kappa t + \frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\log\sqrt{-g}\right) = -\kappa T\,. \tag{88}$$

The combination of these two equations gives the problematic condition (82). Upon inspection of eqs. (87)–(88), one sees that condition (82) would change to condition (85) if, instead of eqs. (87)–(88), one had (cf. Einstein 1915d, eqs. 10 and 9, respectively):

$$g^{\alpha\beta}_{\alpha\beta} - \kappa(t + T) = 0\,, \tag{89}$$

$$g^{\alpha\beta}_{\alpha\beta} - \kappa(t + T) + \frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\log\sqrt{-g}\right) = 0\,. \tag{90}$$

The difference between eq. (88) and eq. (90) is the sign of the term κT. Recall that eq. (88) was obtained by fully contracting the field equations (68). One can change the sign of the term κT in eq. (88) by adding a trace term to the right-hand side of eq. (68):[106]

$$\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right). \tag{91}$$

105 This follows from $\kappa t \equiv \kappa t^{\lambda}_{\lambda} = 2g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\lambda}\Gamma^{\lambda}_{\alpha\nu}$.

106 Contracting $-\kappa(T_{\mu\nu} - (1/2)g_{\mu\nu}T)$ with $g^{\mu\nu}$, one finds κT.

Fully contracting these new field equations and rewriting the resulting equations, one arrives at (cf. eqs. (79)–(81))

$$g^{\alpha\beta}_{\alpha\beta} - g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{\partial}{\partial x^{\alpha}}\left(g^{\alpha\beta}\frac{\partial}{\partial x^{\beta}}\log\sqrt{-g}\right) = \kappa T\,.$$

Using eq. (86) to substitute $-\kappa t$ for the second term on the left-hand side, one arrives at eq. (90).

The new field equations (91) contracted with $g^{\nu\lambda}$ are

$$g^{\nu\lambda}(\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}) = -\kappa\left(T^{\lambda}_{\mu} - \frac{1}{2}\delta^{\lambda}_{\mu}T\right),$$

which can be rewritten as (cf. eq. (75)):

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - \frac{1}{2}\delta^{\lambda}_{\mu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma} = -\kappa(T^{\lambda}_{\mu} + t^{\lambda}_{\mu}) + \frac{1}{2}\delta^{\lambda}_{\mu}T\,.$$

Using eq. (86) to substitute κt for $g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma}$, one arrives at:

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - \frac{1}{2}\delta^{\lambda}_{\mu}\kappa(t + T) = -\kappa(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})\,. \tag{92}$$

Energy-momentum is guaranteed—i.e., $(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})_{,\lambda} = 0$—if the divergence of the left-hand side vanishes:

$$\frac{\partial}{\partial x_{\lambda}}\left[(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - \frac{1}{2}\delta^{\lambda}_{\mu}\kappa(t + T)\right] = 0\,.$$

This can be rewritten (cf. eqs. (76)–(77)) as

$$\frac{\partial}{\partial x^{\mu}}[g^{\alpha\beta}_{\alpha\beta} - \kappa(t + T)] = 0\,.$$

In other words, energy-momentum conservation is guaranteed if the expression in square brackets vanishes, which is just eq. (89). As we noted above, combining eqs. (89) and (90) gives condition (85) which makes it possible to set $\sqrt{-g} = 1$ without any consequences for the value of T. Eq. (90) is a direct consequence of the field equations. Eq. (89), which guarantees energy-momentum conservation, is therefore a consequence of the field equations plus the condition $\sqrt{-g} = 1$. There is no need anymore for the highly speculative assumption that all matter is electromagnetic.

As Norton (1984, 146–147) has emphasized, the addition of the trace term $(1/2)\kappa g_{\mu\nu}T$ to the field equations was not an option for Einstein before the perihelion paper (Einstein 1915c). In the Zurich Notebook, Einstein had briefly considered adding a trace term to field equations based on the Ricci tensor in a weak-field approximation. He had rejected such modified weak-field equations because they do not allow the spatially flat metric, $g_{\mu\nu} = \mathrm{diag}(-1, -1, -1, f(x, y, z))$, which Einstein expected to describe weak static fields.[107] The derivation of the perihelion motion of Mercury freed Einstein from this prejudice. It showed that weak static fields do not have to be spatially flat. This insight was directly related to his use of unimodular coordinates in this calculation (Earman and Janssen 1993, 144–145). If $\sqrt{-g} = 1$ and g_{44} is non-constant, then at least some of the spatial components g_{ij} ($i, j = 1, 2, 3$) must be non-constant as well ($g_{4i} = g_{i4} = 0$ for a static field). This removes the objection against adding a trace term. Einstein could now set $\sqrt{-g} = 1$ without committing himself to the electromagnetic world view.

The field equations (91) with the trace term have another feature that strongly recommends them. Compare the field equations (91) in the form of eq. (92) to the field equations (68) of (Einstein 1915a) in the form of eq. (75). Using eq. (86) to substitute κt in the second term on the left-hand side, one can write eq. (75) as:

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - \frac{1}{2}\delta^{\lambda}_{\mu}\kappa t = -\kappa(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})\,. \tag{93}$$

The crucial difference between eq. (92) and eq. (93) is that in eq. (92) the energy-momentum tensor of matter enters the field equations in the exact same way as the energy-momentum pseudo-tensor for the gravitational field, whereas in eq. (93) it does not. In eq. (93) there is a term $-(1/2)\delta^{\lambda}_{\mu}\kappa T$ missing on the left-hand side. Einstein made this same observation comparing eq. (90) (after setting $\sqrt{-g} = 1$) to its counterpart eq. (88) for the field equations (68) without the trace term:

> Note that our additional [trace] term brings with it that in [eq. 9 of (Einstein 1915d), $g^{\alpha\beta}_{\alpha\beta} - \kappa(t + T) = 0$] the energy tensor of the gravitational field occurs alongside the one for matter in the same way, which is not the case in [the corresponding eq. 21 of (Einstein 1915a), $g^{\alpha\beta}_{\alpha\beta} - \kappa t + \ldots = -\kappa T$].[108]

In the general formalism of (Einstein 1914c), the conditions $B_{\mu} = 0$ guarantee the vanishing of $(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})_{,\lambda}$ and the conditions $S^{\nu}_{\sigma} = 0$ guarantee that $T_{\mu\nu}$ enters the field equations in the same way as $t_{\mu\nu}$ (see secs. 3.1 and 3.2). The conditions $S^{\nu}_{\sigma} = 0$, however, do not hold if the restriction to unimodular transformations or unimodular coordinates is made (see note 88). In his first November paper, Einstein

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107 These considerations can be found on pp. 20L–21R of the Zurich Notebook. See “Commentary …” (this volume), secs. 5.4.3–5.4.4 for a detailed analysis and sec. 5.4.6 for a concise summary. The problem is this. For weak static fields, the field equations with trace term reduce to:

$$\Delta g_{\mu\nu} = \kappa\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right)$$

(with $\eta_{\mu\nu} \equiv \mathrm{diag}(-1, -1, -1, 1)$). For a static mass distribution described by $T_{\mu\nu} = \mathrm{diag}(0, 0, 0, \rho)$ ($T = \rho$), the non-trivial components of these equations are:

$$\Delta g_{11} = \Delta g_{22} = \Delta g_{33} = \Delta g_{44} = \frac{1}{2}\kappa\rho\,.$$

The spatially flat metric $\mathrm{diag}(-1, -1, -1, f)$ is not a solution of these equations.

108 “Man beachte, daß es unser Zusatzglied mit sich bringt daß in (9) der Energietensor des Gravitationsfeldes neben dem der Materie in gleicher Weise auftritt, was in Gleichung (21) a. a. O. nicht der Fall ist” (Einstein 1915d, 846).

made sure that $(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})_{,\lambda} = 0$ holds, but he did not check whether $T_{\mu\nu}$ and $t_{\mu\nu}$ enter the field equations in the same way. If he had, he would have recognized the need for the trace term right away.[109]

Einstein needed to check one more thing in his fourth November paper to make sure that the new field equations (91) with trace term do indeed give energy-momentum conservation in unimodular coordinates. He had to show that $(T^{\lambda}_{\mu} + t^{\lambda}_{\mu})_{,\lambda} = 0$ is equivalent to the energy-momentum balance equation in unimodular coordinates,

$$T^{\lambda}_{\sigma,\lambda} + \frac{1}{2} g^{\mu\nu}_{\sigma} T_{\mu\nu} = 0 . \tag{94}$$

(cf. eqs. (69)–(71)). The question is whether the second term on the left-hand side of this equation can be rewritten as $t^{\lambda}_{\sigma,\lambda}$? The standard procedure for doing this is to replace $T_{\mu\nu}$ in this term by the left-hand side of the field equations (see note 41, eqs. (17)–(19) and eqs. (72)–(73)). In this case, however, the field equations have $T_{\mu\nu} - (1/2) g_{\mu\nu} T$ on the right-hand side rather than simply $T_{\mu\nu}$. It turns out that this does not lead to any complications. In unimodular coordinates, as Einstein (1915d, 846) noted,

$$g^{\mu\nu}_{\sigma} g_{\mu\nu} = -(\log\sqrt{-g})_{,\sigma} = 0 .$$

It follows that the second term of eq. (94) can be written as:

$$\frac{1}{2} g^{\mu\nu}_{\sigma} \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right) .$$

With the help of the field equations (91) this turns into

$$-\frac{1}{2\kappa} g^{\mu\nu}_{\sigma} (\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu} \Gamma^{\beta}_{\alpha\nu}) .$$

As Einstein had already shown in his first November paper (see eqs. (72)–(74)), this expression is equal to the divergence of the gravitational energy-momentum pseudotensor in unimodular coordinates (Einstein 1915d, 846, eq. 8a; 1915a, 785, eq. 20b)

$$\kappa t^{\lambda}_{\sigma} \equiv \frac{1}{2} \delta^{\lambda}_{\sigma} g^{\mu\nu} \Gamma^{\alpha}_{\beta\mu} \Gamma^{\beta}_{\alpha\nu} - g^{\mu\nu} \Gamma^{\alpha}_{\mu\sigma} \Gamma^{\lambda}_{\alpha\nu} ,$$

defined in eq. (74) and used throughout this section and sec. 6. In unimodular coordi-

109 Einstein concisely summarized this part of his struggle to come up with satisfactory field equations in a letter to Besso a little over a month later: “The first paper [Einstein 1915a] along with the addendum [Einstein 1915b] still suffers from the problem that the term $(1/2)\kappa g_{\mu\nu} T$ is missing on the right-hand side; hence the postulate $T = 0$. Obviously, things have to be done as in the last paper [Einstein 1915d], in which case there is no condition anymore on the structure of matter” (“Die erste Abhandlung samt dem Nachtrag krankt noch daran dass auf der rechten Seite das Glied $(1/2)\kappa g_{\mu\nu} T$ fehlt; daher das Postulat $T = 0$. Natürlich muss die Sache gemäss der letzten Arbeit gemacht werden, wobei sich über die Struktur der Materie keine Bedingung mehr ergibt.” Einstein to Michele Besso, 3 January 1916 [CPAE 8, Doc. 178]).

nates, the field equations (91) of (Einstein 1915d) thus satisfy all requirements needed for energy-momentum conservation:

(1) $T_{\mu\nu}$ and $t_{\mu\nu}$ enter the field equations in the exact same way;

(2) The field equations guarantee the vanishing of the divergence of $T_{\mu\nu} + t_{\mu\nu}$;

(3) The divergence of $t_{\mu\nu}$ is equal to the gravitational force density.

As with the field equations (84) of the "Addendum" (Einstein 1915b), Einstein looked upon the field equations (91),

$$R_{im} = -\kappa\left(T_{im} - \frac{1}{2}g_{im}T\right),$$

where R_{im} is the November tensor (Einstein 1915d, 845, eq. 6), as generally-covariant field equations expressed in unimodular coordinates. The corresponding generally-covariant equations are the Einstein field equations,

$$G_{im} = -\kappa\left(T_{im} - \frac{1}{2}g_{im}T\right), \tag{95}$$

where G_{im} is the full Ricci tensor (ibid., eq. 2a). As in (Einstein 1915b), he tacitly assumed that these equations would guarantee energy-momentum conservation in arbitrary coordinates.

To conclude our analysis of Einstein's four papers of November 1915, we summarize what we see as the four key steps in the transition from the "Entwurf" field equations to the Einstein field equations.[110] The first step was the redefinition of the components of the gravitational field which led Einstein back to field equations invariant under unimodular transformations that he had considered and rejected three years earlier in the Zurich Notebook. The second step was to rewrite the four conditions that in conjunction with the field equations guarantee energy-momentum conservation as one condition on g, the determinant of the metric, to reflect the connection between energy-momentum conservation and the covariance of the field equations. This made it possible for Einstein to start using coordinate conditions in the modern sense. The third step was to recognize that the theory could be tweaked to turn the one condition on g into the condition $\sqrt{-g} = 1$ for unimodular coordinates. This made it possible to look upon the new field equations as generally-covariant equations expressed in unimodular coordinates. This is what Einstein had to show for his brief dalliance with the electromagnetic world view. The fourth and final step was to recognize that energy-momentum conservation dictates that such tweaking be done in a specific way, namely through adding a term with the trace of the energy-momentum tensor for matter to the field equations. The perihelion paper (Einstein

110 The first two steps were made in (Einstein 1915a) and are discussed in secs. 5 and 6, respectively; the last two were made in (Einstein 1915b) and (Einstein 1915d), respectively, and are both discussed in this section, sec. 7.

1915c) was important in this context in that it freed Einstein from his prejudice about the form of the metric for weak static fields which he had found to be incompatible with such a trace term in the Zurich Notebook. We reiterate that this whole chain of reasoning was set in motion by replacing definition (53) of the components of the gravitational field, the "fateful prejudice," by definition (54), "the key to the solution." In all of this Einstein relied heavily on the general variational formalism of (Einstein 1914c). The exact expressions, relations, and conditions given by this formalism could not be used because of the restriction to unimodular transformations and unimodular coordinates in the papers of November 1915, but the insights encoded in the formalism were Einstein's main guide in taking steps one, two, and four.

8. THE 1916 REVIEW ARTICLE: THE ARGUMENT OF NOVEMBER 1915 STREAMLINED

Einstein's argument in (Einstein 1915d) for adding a trace term on the right-side of the field equations proved difficult to follow even for those most supportive of his efforts, such as his Leyden colleagues Paul Ehrenfest and H. A. Lorentz. Although Einstein claimed in the introduction of (Einstein 1915d) that the paper was self-contained, it in fact relied heavily on (Einstein 1915a) in its justification of the trace term. The relevant part of (Einstein 1915a) in turn relied heavily on the exposition of the "Entwurf" theory in (Einstein 1914c). In early 1916, in a letter to Ehrenfest, Einstein produced a self-contained version of the argument leading to the trace term and the Einstein field equations of the fourth November communication without the detour through the various discarded field equations preceding them. This letter became the blueprint for the part on field equations and energy-momentum conservation in (Einstein 1916a), the first systematic self-contained exposition of the new theory.

It was clear to Einstein that the field equations of his last communication of November 1915 met all requirements that he had imposed on such equations and that no further changes would be needed. Given the rapid succession of different field equations during that one month, however, it is understandable that this was not so clear to his readers. Even those most supportive of Einstein's efforts, such as the Leyden physicists Paul Ehrenfest and H. A. Lorentz, had difficulties following the argument.

Einstein himself best described the impression that the flurry of papers of November 1915 must have made on his colleagues. Knowing that the final result was correct and fully aware of the monumental character of his achievement, Einstein could afford to poke fun at the chaotic way in which victory had at long last been achieved. "It's convenient with that fellow Einstein," he wrote to Ehrenfest, "every year he retracts what he wrote the year before."[111] With similar self-deprecation, he told Sommerfeld: "Unfortunately I have immortalized my final errors in the academy-papers [Einstein 1915a, b] that I can soon send you."[112] When he did send the papers a week and a half later, he urged Sommerfeld to study them carefully despite the fact "that, as you are reading, the final part of the battle for the field equations unfolds right in front of your eyes."[113]

As was his habit, Ehrenfest pestered his friend Einstein with questions about the new theory.[114] Lorentz, who had already filled an uncounted number of pages with calculations on the "Entwurf" theory and had cast the theory in Lagrangian form (Lorentz 1915), immediately went to work on the new theory and sent Einstein three letters with comments and queries.[115] One topic of discussion was the hole argument, which Einstein had silently and unceremoniously dropped upon his return to general covariance in November 1915.[116] For our purposes in this paper, the interesting part of the discussion concerns the relation between the field equations and energy-momentum conservation and the necessity of the trace term. At one point in his correspondence with Ehrenfest, Einstein refers to "the warrant demanded by you for the inevitability of the additional term $-(1/2)g_{im}T$."[117]

Ehrenfest's obstinacy paid off. Einstein finally broke down and sent him a lengthy self-contained version of the argument that before had to be pieced together from the papers of November 1915. As Einstein promised at the beginning of the letter: "I shall not rely on the [November 1915] papers at all but show you all the calculations."[118] After delivering on this promise, Einstein closed the letter saying:

> I assume you will have no further difficulty. Show the thing to Lorentz too, who also has not yet appreciated the necessity of the structure of the right-hand side of the field equations. Could you do me a favor and send these sheets back to me as I do not have these things so neatly in one place anywhere else.[119]

Ehrenfest presumably obliged. The letter reads like the blueprint for the sections on

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111 "Es ist bequem mit dem Einstein. Jedes Jahr widerruft er, was er das vorige Jahr geschrieben hat." Einstein to Paul Ehrenfest, 26 December 1915 (CPAE 8, Doc. 173). With this comment, Einstein prefaced his retraction of the hole argument (see sec. 4 of "What did Einstein know …" [this volume]).

112 "Die letzten Irrtümer in diesem Kampfe habe ich leider in den Akademie-Arbeiten, die ich Ihnen bald senden kann, verevigt [sic]." Einstein to Arnold Sommerfeld, 28 November 1915 (CPAE 8, Doc. 153).

113 "… dass sich beim Lesen der letzte Teil des Kampfes um die Feldgleichungen vor Ihren Augen abspielt." Einstein to Arnold Sommerfeld, 9 December 1915 (CPAE 8, Doc. 161).

114 For discussion of the relationship between Einstein and Ehrenfest, see Klein 1970, Ch. 12.

115 This can be inferred from Einstein to H. A. Lorentz, 17 January 1916 (CPAE 8, Doc. 183). Unfortunately, none of the letters from Ehrenfest and Lorentz to Einstein of this period (late 1915–early 1916) seem to have been preserved. For a discussion of the three-way correspondence between Einstein, Lorentz and Ehrenfest in this period, see Kox 1988.

116 For discussion of the hole argument and its replacement, the point-coincidence argument, and references to the extensive literature on these topics, see sec. 4 of "What did Einstein know …" (this volume).

117 "die von Dir verlangte Gewähr der „Zwangläufigkeit" für das Zusatzglied $-(1/2)g_{im}T$." Einstein to Paul Ehrenfest, 17 January 1916 (CPAE 8, Doc. 182).

118 "Ich stütze mich gar nicht auf die Arbeiten, sondern rechne Dir alles vor." Einstein to Paul Ehrenfest, 24 January 1916 or later (CPAE 8, Doc. 185).

119 "Du wirst nun wohl keine Schwierigkeit mehr finden. Zeige die Sache auch Lorentz, der die Notwendigkeit der Struktur der rechten Seite der Feldgleichungen auch noch nicht empfindet. Es wäre mir lieb, wenn Du mir diese Blätter dann wieder zurückgäbest, weil ich die Sachen sonst nirgends so hübsch beisammen habe." Ibid.

the field equations and energy-momentum conservation in (Einstein 1916a), the first systematic exposition of general relativity, sent to Willy Wien, the editor of *Annalen der Physik*, in March 1916[120] and published in May of that year.

As in his papers of November 1915 and in the letter to Ehrenfest, Einstein used unimodular coordinates in this paper. He started with the November Lagrangian

$$H = g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} \tag{96}$$

(Einstein 1916a, 804, eq. 47a; see eq. (64) above with $\mathrm{L} = H$ rather than $-H$). The Euler-Lagrange equations,

$$\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial H}{\partial g^{\mu\nu}} = 0 \tag{97}$$

(ibid., 805, eq. 47b), for this Lagrangian are:

$$\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = 0 \tag{98}$$

(ibid., 803, eq. 47; see eqs. (64)–(68) with $T_{\mu\nu} = 0$). These are the vacuum field equations. The question is how to generalize these equations in the presence of matter. To this end, Einstein rewrote the vacuum field equations in terms of the gravitational energy-momentum pseudo-tensor $t_{\mu\nu}$. He then added the energy-momentum tensor for matter $T_{\mu\nu}$ in such a way that it enters the field equations in the exact same way as $t_{\mu\nu}$. This strategy originated in the letter to Ehrenfest. After writing down the field equations in the form of eq. (92) (eq. 8 in the letter), Einstein wrote: "This equation is interesting because it shows that the origin of the gravitational [field] lines is determined solely by the sum $T^{\nu}_{\sigma} + t^{\nu}_{\sigma}$, as one has to expect."[121]

To find $t_{\mu\nu}$, Einstein contracted the left-hand side of the eq. (97) with $g^{\mu\nu}_{\sigma}$

$$g^{\mu\nu}_{\sigma}\left(\frac{\partial}{\partial x^{\alpha}}\left(\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}}\right) - \frac{\partial H}{\partial g^{\mu\nu}}\right) = 0\,. \tag{99}$$

Since he did not have the field equations in the presence of matter yet, Einstein could not give the usual rationale for this move. Using eq. (91), the Einstein field equations in the presence of matter in unimodular coordinates, one can rewrite the left-hand side of eq. (99) as $\kappa g^{\mu\nu}_{\sigma}(T_{\mu\nu} - (1/2)g_{\mu\nu}T)$. As we saw in sec. 7, in unimodular coordinates $g^{\mu\nu}_{\sigma}g_{\mu\nu} = 0$, so this is equal to $\kappa g^{\mu\nu}_{\sigma}T_{\mu\nu}$, which is 2κ times the gravitational force density. By writing the left hand side of eq. (99) as a divergence, Einstein could thus express the gravitational force density as the divergence of gravitational energy-momentum density. As we have seen, this was Einstein's stan-

120 Einstein to Wilhelm Wien, 18 March 1916 (CPAE 8, Doc. 196). Einstein had told Wien in February that he was in the process of writing this paper (Einstein to Wilhelm Wien, 28 February 1916 [CPAE 8, Doc. 196]).

121 "Diese Gleichung ist interessant, weil sie zeigt, dass das Entspringen der Gravitationslinien allein durch die Summe $T^{\nu}_{\sigma} + t^{\nu}_{\sigma}$ bestimmt ist, wie man ja auch erwarten muss." Einstein to Paul Ehrenfest, 24 January 1916 or later (CPAE 8, Doc. 185).

dard procedure for introducing $t_{\mu\nu}$ (see note 41, eqs. (17)–(19), eqs. (72)–(73), and the derivation following eq. (94)).

Eq. (99) can be rewritten as $t^{\alpha}_{\sigma,\alpha} = 0$, if t^{α}_{σ} is defined as[122]

$$-2\kappa t^{\alpha}_{\sigma} = g^{\mu\nu}_{\sigma}\frac{\partial H}{\partial g^{\mu\nu}_{\alpha}} - \delta^{\alpha}_{\sigma}H \tag{100}$$

(Einstein 1916a, 805, eq. 49). Substituting eq. (67) (with $\mathrm{L} = H$) and eq. (96) into eq. (100), one finds (cf. eq. (74))

$$\kappa t^{\alpha}_{\sigma} = \frac{1}{2}\delta^{\alpha}_{\sigma}g^{\mu\nu}\Gamma^{\beta}_{\beta\mu}\Gamma^{\beta}_{\rho\nu} - g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\sigma} \tag{101}$$

(ibid., 806, eq. 50).

Einstein now used t^{μ}_{ν} to rewrite the field equations (98). The trace of the pseudo-tensor is (see eq. (86)):

$$\kappa t = g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}\,.$$

Eq. (101) can thus be rewritten as

$$\kappa\left(t^{\alpha}_{\sigma} - \frac{1}{2}\delta^{\alpha}_{\sigma}t\right) = -g^{\mu\nu}\Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\sigma}\,. \tag{102}$$

The contraction of the vacuum field equations (98) with $g^{\nu\sigma}$,

$$g^{\nu\sigma}(\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha}) = 0\,,$$

can be rewritten as (see note 89):

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\rho\tau}\Gamma^{\sigma}_{\rho\beta}\Gamma^{\beta}_{\tau\mu} = 0\,.$$

Using eq. (102) for the second term, one can thus write the vacuum field equations in unimodular coordinates as

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} = -\kappa\left(t^{\sigma}_{\mu} - \frac{1}{2}\delta^{\sigma}_{\mu}t\right) \tag{103}$$

(ibid., 806, eq. 51).

On the argument that $T_{\mu\nu}$ should enter the field equations in the exact same way as $t_{\mu\nu}$, Einstein generalized the vacuum equations to

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} = -\kappa\left([t^{\sigma}_{\mu} + T^{\sigma}_{\mu}] - \frac{1}{2}\delta^{\sigma}_{\mu}[t + T]\right) \tag{104}$$

122 Einstein (1916a, 805) added a footnote saying: “The reason for the introduction of the factor -2κ will become clear later” (“Der Grund der Einführung des Faktors -2κ wird später deutlich werden.”). He is referring to the generalization of the vacuum field equations (103) to the field equations (104) in the presence of matter in sec. 16 and to the discussion of energy-momentum conservation in secs. 17–18 of his paper (ibid., 807–810).

in the presence of matter (ibid., 807, eq. 52). Since eq. (103) is just an alternative way of writing the vacuum field equations contracted with $g^{\nu\sigma}$, eq. (104) is equivalent to

$$\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\mu\beta}\Gamma^{\beta}_{\nu\alpha} = -\kappa\Big(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\Big)$$

(ibid., 808, eq. 53; cf. eqs. (91)). These equations are more easily recognized as the generally-covariant Einstein field equations (95) in unimodular coordinates.

In the next section of his paper, Einstein (1916a, sec. 17) showed that energy-momentum conservation in the form $(T^{\alpha}_{\sigma} + t^{\alpha}_{\sigma})_{,\alpha} = 0$ is a direct consequence of the field equations (104). Fully contracting eq. (104), one finds

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\sigma\nu})_{,\alpha} = \kappa(t + T), \tag{105}$$

with the help of which eq. (104) itself can be rewritten as

$$\frac{\partial}{\partial x^{\alpha}}\Big(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu} - \frac{1}{2}\delta^{\sigma}_{\mu}[g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}]\Big) = -\kappa(t^{\sigma}_{\mu} + T^{\sigma}_{\mu}).$$

The field equations thus guarantee energy-momentum conservation if

$$B_{\mu} \equiv \frac{\partial^2}{\partial x^{\sigma}\partial x^{\alpha}}\Big(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu} - \frac{1}{2}\delta^{\sigma}_{\mu}[g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta}]\Big) = 0. \tag{106}$$

This equation, it turns out, is an identity. The first term can be rewritten as[123]

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu})_{,\alpha\sigma} = \frac{1}{2}g^{\alpha\beta}_{\alpha\beta\mu};$$

the second as minus this same expression.[124] Eq. (106) gives the contracted Bianchi

123 Using the definition (54) of $\Gamma^{\alpha}_{\mu\nu}$ as minus the Christoffel symbols, one can write

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu})_{,\alpha\sigma} = -\frac{1}{2}(g^{\sigma\beta}g^{\alpha\lambda}(g_{\lambda\mu,\beta} + g_{\lambda\beta,\mu} - g_{\mu\beta,\lambda}))_{,\alpha\sigma}$$

The combination of the first and the third term in the innermost parentheses are anti-symmetric in λ and β. They are contracted with a quantity symmetric in these same indices (cf. note 92 above).

$$(g^{\sigma\beta}g^{\alpha\lambda}(\ldots))_{,\alpha\sigma} = (g^{\alpha\beta}g^{\sigma\lambda}(\ldots))_{,\sigma\alpha} = (g^{\sigma\lambda}g^{\alpha\beta}(\ldots))_{,\alpha\sigma}$$

The expression above thus reduces to

$$(g^{\nu\sigma}\Gamma^{\alpha}_{\mu\nu})_{,\alpha\sigma} = -\frac{1}{2}(g^{\sigma\beta}g^{\alpha\lambda}g_{\lambda\beta,\mu})_{,\alpha\sigma} = \frac{1}{2}g^{\alpha\sigma}_{\mu\alpha\sigma}.$$

124 Using eq. (54), one can write

$$-\frac{1}{2}(\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta})_{,\alpha\sigma} = -\frac{1}{2}(g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta})_{,\alpha\mu} = \frac{1}{4}(g^{\lambda\beta}g^{\alpha\delta}(g_{\delta\lambda,\beta} + g_{\delta\beta,\lambda} - g_{\lambda\beta,\delta}))_{,\alpha\mu}.$$

Since $g^{\lambda\beta}g_{\lambda\beta,\delta} = (\log\sqrt{-g})_{,\delta}$ (Einstein 1916a, 796, eq. 29) vanishes for unimodular coordinates, the expression above reduces to

$$-\frac{1}{2}(\delta^{\sigma}_{\mu}g^{\lambda\beta}\Gamma^{\alpha}_{\lambda\beta})_{,\alpha\sigma} = \frac{1}{2}(g^{\lambda\beta}g^{\alpha\delta}g_{\delta\lambda,\beta})_{,\alpha\mu} = -\frac{1}{2}g^{\alpha\beta}_{\beta\alpha\mu}.$$

identities $(R^{\mu\nu} - (1/2)g^{\mu\nu}R)_{;\nu} = 0$ in unimodular coordinates.

Einstein (1916a, 809) finally showed that energy-momentum conservation in the form $(T^{\alpha}_{\sigma} + t^{\alpha}_{\sigma})_{,\alpha} = 0$ is equivalent to the energy-momentum balance equation $T^{\alpha}_{\sigma,\alpha} + (1/2)g^{\mu\nu}_{\sigma}T_{\mu\nu} = 0$ (see the derivation following eq. (94)).

Three short sections of the review paper (Einstein 1916a, part C, secs. 15–18, pp. 804–810) thus provided a streamlined version of an argument that had been hard to piece together from the four papers of November 1915 even for the likes of Lorentz and Ehrenfest.

9. FROM THE NOVEMBER LAGRANGIAN TO THE RIEMANN SCALAR: GENERAL COVARIANCE AND ENERGY-MOMENTUM CONSERVATION

Both in the papers of November 1915 and in the review article that following March (Einstein 1916a), Einstein used the November Lagrangian—i.e., the "Entwurf" Lagrangian with the components of the gravitational field redefined as minus the Christoffel symbols—to derive the gravitational part of the Einstein field equations in unimodular coordinates. The use of unimodular coordinates clearly brings out the relation between the new field equations and the old "Entwurf" field equations (see the appendix), but it complicates the use of the general formalism of (Einstein 1914c). Most seriously affected by this problem is the discussion of energy-momentum conservation. Einstein was able to show that the field equations guarantee energy-momentum calculation in unimodular coordinates, but he did not prove that this will also be true in arbitrary coordinates. He also did not make the connection between energy-momentum conservation and the covariance of the field equations. In (Einstein 1916c) the November Lagrangian is replaced by a Lagrangian extracted from the Riemann scalar. Applying the formalism of (Einstein 1914c), Einstein showed that energy-momentum conservation holds in arbitrary coordinates and that it follows directly from the general covariance of the Einstein field equations. This result is expressed in two sets of conditions, $S^{\nu}_{\sigma} = 0$ *and* $B_{\mu} = 0$. *The latter are just the contracted Bianchi identities.*

Shortly after the triumphs of November 1915, Einstein acknowledged the desirability of deriving (the gravitational part of) the generally-covariant form of the field equations from a variational principle. In the November 1915 papers, as we have seen, he had only done so in unimodular coordinates (Einstein 1915a, 784). He realized that the generally-covariant Lagrangian would have to come from the Riemann curvature scalar. He also realized that terms with second-order derivatives of the metric in the Riemann scalar, which would lead to terms with third-order derivatives in the field equations, could be eliminated from the action through partial integration. He concluded that the effective Lagrangian had to be

$$L = \sqrt{-g}[g^{\sigma\tau}\Gamma^{\beta}_{\sigma\alpha}\Gamma^{\alpha}_{\tau\beta} - g^{\alpha\beta}\Gamma^{\sigma}_{\alpha\beta}\Gamma^{\rho}_{\sigma\rho}].$$

All this can be found in a letter to Lorentz of January 17, 1916. Einstein wrote that he had only gone through the calculation once, but the expression above is actually correct. He also told Lorentz that he had not attempted to derive the corresponding Euler-Lagrange equations: "The calculation of $\partial L / \partial g^{\mu\nu}$ and $\partial L / \partial g_{\sigma}^{\mu\nu}$, however, is rather cumbersome, at least with my limited proficiency in calculating."[125]

When he wrote the review article (Einstein 1916a) less than two months later, he apparently still did not have the stomach for this cumbersome though straightforward calculation. As we have seen in sec. 8, both the presentation of the field equations and the discussion of energy-momentum conservation in secs. 14–18 of (Einstein 1916a) are in terms of unimodular coordinates.

Einstein may originally have planned to cover this material in arbitrary coordinates. This is suggested by a manuscript for an ultimately discarded five-page appendix to the review article (CPAE 6, Doc. 31). At the top of the first page of the manuscript, we find "§14" which was subsequently deleted and replaced by "Appendix: Formulation of the theory based on a variational principle."[126] Sec. 14 is the first of the five sections in (Einstein 1916a) on the field equations and energy-momentum conservation. In the manuscript under consideration here, Einstein gave a variational derivation of the field equations in arbitrary coordinates along the lines sketched in the letter to Lorentz discussed above. He still did not explicitly evaluate the Euler-Lagrange equations. But he did write down the quantities S_{σ}^{ν} and B_{σ} of the general formalism of (Einstein 1914c) for the effective Lagrangian of the new theory (now denoted by $\mathfrak{G}$). He pointed out that the general covariance of the Riemann scalar guarantees that these quantities vanish identically. He did not mention that this automatically implies energy-momentum conservation (see sec. 3.2 and 3.3). This might simply be because Einstein did not bother to finish this manuscript once he had decided to rewrite sec. 14 and the remainder of part C of his review article in unimodular coordinates. The original generally-covariant treatment was relegated to an appendix, which ultimately did not make it into the published paper. Einstein returned to it a few months later, revised and completed the manuscript, and submitted it to the Prussian Academy on October 26, 1916. This paper, (Einstein 1916c), will be the main focus of this section. But first we discuss some of Einstein's pronouncements on the topic in the intervening months.

Ehrenfest must have taken Einstein to task for using unimodular coordinates in the crucial sections of the review article (Einstein 1916a). In May 1916, shortly after the article was published, Einstein wrote to his friend in Leyden defensively: "My specialization of the coordinate system is not *just* based on laziness."[127] Did Einstein

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125 "Die Berechnung von $\partial L / \partial g^{\mu\nu}$ und $\partial L / \partial g_{\sigma}^{\mu\nu}$ ist aber ziemlich beschwerlich, wenigstens bei meiner geringen Sicherheit im Rechnen." Einstein to H. A. Lorentz, 17 January 1916 (CPAE 8, Doc. 183). He explicitly said he had not done the calculation in another letter to Lorentz two days later (Einstein to H. A. Lorentz, 19 January 1916 [CPAE 8, Doc. 184]).

126 "Anhang: Darstellung der Theorie ausgehend von einem Variationsprinzip." (CPAE 6, Doc. 31, [p. 1]).

127 "Meine Spezialisierung des Bezugssystems beruht nicht *nur* auf Faulheit." Einstein to Paul Ehrenfest, 24 May 1916 (CPAE 8, Doc. 220).

have some reason to believe that the choice of unimodular coordinates was not just convenient but physically meaningful?[128] At the time of this letter to Ehrenfest, it may not have been more than an inkling, but a month later Einstein actually published an argument purporting to show that unimodular coordinates are indeed physically privileged (Einstein 1916b).

Given that general relativity had been developed in analogy with electrodynamics (see sec. 3.1 and 3.2), it was only natural for Einstein to explore the possibility of gravitational waves in his theory. This is what he did in (Einstein 1916b). He found three types of waves, two of which curiously do not transport energy (Einstein 1916b, 693). In an addendum to the paper, Einstein noted that these spurious waves can be eliminated by choosing unimodular coordinates. This, he concluded, shows that the choice of unimodular coordinates has “a deep physical justification.”[129] He also rehearsed this argument in a letter to De Sitter, another member of the Leyden group around Lorentz and Ehrenfest working on relativity.[130]

Two letters from Leyden to Berlin the following year suggest that, in late 1917, Einstein still believed that unimodular coordinates have a special status. The author of these letters was Gunnar Nordström, who was in Leyden on a three-year fellowship (CPAE 8, Doc. 112, note 3). Nordström had a hard time convincing Einstein that in unimodular coordinates the gravitational field of the sun carries no energy.[131] Nordström also caught an error in (Einstein 1916b), which prompted Einstein to publish a corrected version of his 1916 paper on gravitational waves (Einstein 1918a). He now used a different argument to eliminate the spurious gravitational waves, one that makes no mention of unimodular coordinates (ibid., 160–161). By the time Gustav Mie, in his efforts to convince Einstein of the need for special coordinates, reminded him of the original argument, Einstein had abandoned the notion of privileged coordinates, unimodular or otherwise, altogether.[132]

Immediately following the defensive passage from the letter to Ehrenfest of May 1916 quoted above (see note 127), Einstein promised: “At some point I may present the matter without such specialization [of the coordinates], along the lines of [Lorentz 1915].”[133] Given Einstein’s views at the time that unimodular coordinates were special, such a fully generally-covariant presentation of the theory was probably

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128 In December 1915, Einstein had called the choice of unimodular coordinates “epistemologically meaningless” (“erkenntnistheoretisch ohne Bedeutung.” Einstein to Moritz Schlick, 14 December 1915 [CPAE 8, Doc. 165]).

129 “eine tiefe physikalische Berechtigung” (Einstein 1916b, 696).

130 Einstein to Willem de Sitter, 22 June 1916 (CPAE 6, Doc. 32). The work of the Leyden group is described in (Kox 1992).

131 Gunnar Nordström to Einstein, 22–28 September 1917, 23 October 1917 (CPAE 8, Docs. 382, 393)

132 Gustav Mie to Einstein, 6 May 1918 (CPAE 8, Doc. 532). For brief discussions of the episode described in the last two paragraphs, see CPAE 8, li-lii, and CPAE 7, xxv.

133 “Vielleicht werde ich die Sache auch einmal ohne die Spezialisierung darstellen, so wie Lorentz in seiner Arbeit.”

not a matter of great urgency to him. He was nonetheless forced to keep thinking about the issue, not by Ehrenfest this time but by a new correspondent.

In June 1916, Théophile de Donder, professor of mathematical physics in Brussels, respectfully informed Einstein that the latter's expression for the gravitational energy-momentum pseudo-tensor was wrong (Théophile de Donder to Einstein, 27 June 1916 [CPAE 8, Doc. 228]). An exchange of letters across enemy lines ensued, mercifully cut short a little over a month later by exhaustion on the Belgian side. De Donder began what would turn out to be the last letter of this testy correspondence with the announcement of a truce of sorts: "the extensive research and innumerable calculations that I have devoted to your theory have forced me to take some rest for a few weeks."[134] Einstein probably read this with a sigh of relief. Even though De Donder's missives were ostensibly about clarifying the relation of his own work to Einstein's, it is hard not to get the impression that De Donder's ulterior motive was to have Einstein concede priority for at least part of general relativity to his Belgian colleague.[135] If this was indeed De Donder's hidden agenda, he must have been bitterly disappointed by the letters from Berlin. The way Einstein saw it, De Donder had simply overlooked that the expressions whose correctness he was contesting only held in unimodular coordinates. In Einstein's last contribution to the debate, he showed how one would obtain the expression for the gravitational energy-momentum pseudo-tensor without choosing unimodular coordinates. The formula that Einstein gives is

$$\mathfrak{t}_\sigma^\nu = -\frac{1}{2\kappa}\left(\frac{\partial L^*}{\partial g_\nu^{\alpha\beta}} g_\sigma^{\alpha\beta} - \delta_\sigma^\nu L^*\right).$$

L^* is Einstein's notation in this letter for the effective Lagrangian extracted from the Riemann scalar.[136] The letter shows that Einstein had no trouble with the continuation of the argument of the appendix to (Einstein 1916a) discussed above.

In the fall of 1916, Einstein finally finished what he had begun in this appendix and published a generally-covariant discussion of the field equations and energy-momentum conservation. The paper, (Einstein 1916c), brings together in a systematic fashion the various elements of this discussion that we encountered piecemeal in the letter to Lorentz, the discarded appendix, and the letter to De Donder. As is acknowledged in the introduction of (Einstein 1916c), both Hilbert (1915) and Lorentz

134 "Les longues recherches et les innombrables calculs que j'ai consacré à vos théories m'obligent à prendre quelques semaines de repos." Théophile de Donder to Einstein, 8 August 1916 (CPAE 8, Doc. 249)

135 The following year, De Donder (1917) claimed priority for the field equations with cosmological constant of (Einstein 1917b). This prompted Einstein to write to Lorentz who had communicated De Donder's paper to the Amsterdam Academy. Clearly embarrassed to bother Lorentz with this matter, Einstein emphasized that it was because of a serious error in De Donder's paper not because of the priority claim that he urged his Dutch colleague to have De Donder publish a correction (Einstein to H. A. Lorentz, 18 December 1917 [CPAE 8, Doc. 413]). We do not know whether Lorentz took up this matter with De Donder. We do know that no correction ever appeared.

136 In terms of the more explicit notation introduced in our discussion of the general formalism of (Einstein 1914c) in sec. 3, $\mathfrak{t}_\sigma^\nu$ would be written as $\mathfrak{t}_\sigma^\nu(L^*, \mathrm{cons})$ (cf. eq. (19)).

(1916a) had already shown how to derive the Einstein field equations from a variational principle without choosing special coordinates.[137] Einstein's own paper owes little or nothing to this earlier work.[138] It follows the relevant sections of (Einstein 1914c) virtually step by step.

Einstein starts from the action

$$\int \mathrm{H} d\tau = \int (\mathrm{G} + \mathrm{M}) d\tau,$$

where G, the gravitational part of the Lagrangian, is the Riemann scalar and M, the Lagrangian for the material part of the system, is left unspecified (Einstein 1916c, 1111–1112).[139] Through partial integration, all terms involving second-order derivatives of the metric can be removed from the integral over G. The gravitational part of the field equations thus follows from the variational principle

$$\delta \int \mathrm{G}^* d\tau = 0, \tag{107}$$

where G^* is the effective Lagrangian we encountered at the beginning of this section

$$\mathrm{G}^* = \sqrt{-g} g^{\mu\nu} \left[\begin{Bmatrix} \beta \\ \mu\alpha \end{Bmatrix} \begin{Bmatrix} \alpha \\ \nu\beta \end{Bmatrix} - \begin{Bmatrix} \alpha \\ \mu\nu \end{Bmatrix} \begin{Bmatrix} \alpha \\ \alpha\beta \end{Bmatrix} \right] \tag{108}$$

(Ibid., 1113, note 2). For $\sqrt{-g} = 1$, this expression for G^* reduces to expression (96) for the Lagrangian L in unimodular coordinates used in (Einstein 1916a).[140]

The field equations are the Euler-Lagrange equations

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137 See (Renn and Stachel 1999) and (Sauer 1999) for discussion of Hilbert's work; and (Janssen 1992) for discussion of Lorentz's work.

138 See note 46 for discussion of how Einstein's variational techniques deviated from the standard techniques of the Göttingen crowd.

139 As Einstein writes in the introduction: "In particular, specific assumptions about the constitution of matter should be kept to a minimum, in contrast especially to Hilbert's presentation" ("Insbesondere sollen über die Konstitution der Materie möglichst wenig spezialisierende Annahmen gemacht werden, im Gegensatz besonders zur Hilbertschen Darstellung." Einstein 1916c, 1111). Following Mie, Hilbert (1915) had endorsed the electromagnetic world view, according to which the matter Lagrangian is a function only of $g^{\mu\nu}$ and of the components A_μ of the electromagnetic four-vector potential and their first-order derivatives. Einstein had tired of this electromagnetic program almost as fast as he had become enamored of it in November 1915. In a footnote to the discarded appendix to (Einstein 1916a), he characterized Hilbert's approach as "not very promising" ("wenig aussichtsvoll"). This phrase was meant for public consumption. He was much more dismissive of Hilbert's work in the letter to Ehrenfest from which we already quoted in notes 127 and 133 (see also note 152 below). And reporting on (Einstein 1916c) to Weyl, Einstein bluntly wrote: "Hilbert's assumption about matter seems infantile to me, in the sense of a child innocent of the deceit of the outside world" ("Der Hilbertsche Ansatz für die Materie erscheint mir kindlich, im Sinne des Kindes, das keine Tükken der Aussenwelt kennt." Einstein to Hermann Weyl, 23 November 1916 [CPAE 8, Doc. 278]).

140 The second term in square brackets in eq. (108) vanishes since $\begin{Bmatrix} \alpha \\ \alpha\beta \end{Bmatrix} = (\lg\sqrt{-g})_{,\beta}$.

$$\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}_\alpha}\right) - \frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}} = \frac{\partial \mathrm{M}}{\partial g^{\mu\nu}} \tag{109}$$

(Einstein 1916c, 1113, eq. 7). Einstein still did not bother to evaluate the functional derivatives $\partial \mathrm{G}^* / \partial g^{\mu\nu}_\alpha$ and $\partial \mathrm{G}^* / \partial g^{\mu\nu}$ to show that the left-hand side reproduces the Einstein tensor (or rather, the corresponding tensor density). The right-hand side gives minus the energy-momentum tensor density for matter:[141]

$$\mathrm{T}_{\mu\nu} \equiv -\frac{\partial \mathrm{M}}{\partial g^{\mu\nu}} \tag{110}$$

(Einstein 1916c, 1115, eq. 19).

Substituting this definition into eq. (109) and contracting the resulting equations with $g^{\mu\sigma}$, one arrives at:

$$g^{\mu\sigma}\left(\frac{\partial}{\partial x^\alpha}\left(\frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}_\alpha}\right) - \frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}}\right) = -\mathrm{T}^\sigma_\nu .$$

As we have seen in sec. 3.1 (eqs. (11)–(13)), this equation can be rewritten as

$$\frac{\partial}{\partial x^\alpha}\left(g^{\mu\sigma}\frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}_\alpha}\right) = -(\mathrm{T}^\sigma_\nu + \mathrm{t}^\sigma_\nu) \tag{111}$$

(ibid., eq. 18), if t^σ_ν, the gravitational energy-momentum pseudo-tensor, is defined as:

$$\mathrm{t}^\sigma_\nu \equiv -g^{\mu\sigma}_\alpha\frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}_\alpha} - g^{\mu\sigma}\frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}} \tag{112}$$

(ibid., eq. 20, first part).

Energy-momentum conservation in the form

$$(\mathrm{T}^\sigma_\nu + \mathrm{t}^\sigma_\nu)_{,\sigma} = 0 \tag{113}$$

(ibid., eq. 21) is guaranteed through eq. (111) if

$$B_\nu \equiv \frac{\partial^2}{\partial x^\sigma \partial x^\alpha}\left(g^{\mu\sigma}\frac{\partial \mathrm{G}^*}{\partial g^{\mu\nu}_\alpha}\right) = 0 \tag{114}$$

(ibid., eq. 17; Einstein does not use the notation B_μ in this paper).

Eq. (113) is equivalent to energy-momentum conservation in the form $T^{\mu\nu}{}_{;\nu} = 0$, or, equivalently (see eqs. (14)–(17) and note 35),

$$\mathrm{T}^\sigma_{\nu,\sigma} + \frac{1}{2}g^{\alpha\beta}_\nu \mathrm{T}_{\alpha\beta} = 0$$

(ibid., 1116, eq. 22), if

141 Factors of κ, the gravitational constant, are not to be found in (Einstein 1916c). Presumably, they are absorbed into the Lagrangians G^* and M for matter and gravitational field.

$$\mathrm{t}^{\sigma}_{\nu,\sigma} = \frac{1}{2} g^{\alpha\beta}_{\nu} \mathrm{T}_{\alpha\beta} .$$

Using definition (110) of $\mathrm{T}_{\alpha\beta}$ and the field eqs. (109), one can rewrite this as

$$\mathrm{t}^{\sigma}_{\nu,\sigma} = -\frac{1}{2} g^{\alpha\beta}_{\nu}\left(\frac{\partial}{\partial x^{\lambda}}\left(\frac{\partial \mathrm{G}^*}{\partial g^{\alpha\beta}_{\lambda}}\right) - \frac{\partial \mathrm{G}^*}{\partial g^{\alpha\beta}}\right) .$$

As we have seen in sec. 3.2 (eqs. (18)–(19)), this equation holds, if $\mathrm{t}^{\sigma}_{\nu}$ is defined as

$$\mathrm{t}^{\sigma}_{\nu} \equiv \frac{1}{2}\left(\delta^{\sigma}_{\nu}\mathrm{G}^* - g^{\alpha\beta}_{\nu}\frac{\partial \mathrm{G}^*}{\partial g^{\alpha\beta}_{\sigma}}\right) \tag{115}$$

(ibid., 1115, eq. 20, second part).

Compatibility of the definitions (112) and (115) of $\mathrm{t}^{\sigma}_{\nu}$ requires that[142]

$$S^{\nu}_{\sigma} \equiv 2 g^{\beta\nu}_{\alpha}\frac{\partial \mathrm{G}^*}{\partial g^{\beta\sigma}_{\alpha}} + 2 g^{\alpha\nu}\frac{\partial \mathrm{G}^*}{\partial g^{\alpha\sigma}} + \delta^{\nu}_{\sigma}\mathrm{G}^* - g^{\alpha\beta}_{\sigma}\frac{\partial \mathrm{G}^*}{\partial g^{\alpha\beta}_{\nu}} = 0 . \tag{116}$$

Energy-momentum conservation thus requires both eq. (114) and eq. (116):

$$B_{\sigma} = 0, \quad S^{\nu}_{\sigma} = 0 \tag{117}$$

(see sec. 3.2, eq. (24)).

Before these considerations of energy-momentum conservation Einstein (1916c, 1114–1115) has already shown that both equations are satisfied identically for G^* as a consequence of the general covariance of the action in eq. (107).[143] Consider a coordinate transformation $x'^{\mu} = x^{\mu} + \Delta x^{\mu}$, where the Δx^{μ} are chosen such that they vanish outside of some arbitrarily chosen region of space-time. Since the integral over G^* only differs by surface terms from the integral over the Riemann scalar G, the invariance of the latter under coordinate transformations $x'^{\mu} = x^{\mu} + \Delta x^{\mu}$ implies that the former is invariant under such transformations as well. Hence,

$$0 = \Delta\int \mathrm{G}^* d\tau = \int \Delta\left(\frac{\mathrm{G}^*}{\sqrt{-g}}\right)\sqrt{-g}\, d\tau \tag{118}$$

(the second step is justified because $\sqrt{-g}\,d\tau$ is an invariant volume element). The integrand is the sum of two terms:

$$\sqrt{-g}\,\Delta\left(\frac{\mathrm{G}^*}{\sqrt{-g}}\right) = \Delta \mathrm{G}^* + \sqrt{-g}\,\Delta\left(\frac{1}{\sqrt{-g}}\right)\mathrm{G}^* . \tag{119}$$

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142 In the more explicit notation of sec. 3, eq. (112) defines $\mathrm{t}^{\sigma}_{\nu}(\mathrm{G}^*, \text{source})$ and eq. (115) defines $\mathrm{t}^{\sigma}_{\nu}(\mathrm{G}^*, \text{cons})$. Compatibility requires that $S^{\sigma}_{\nu}(\mathrm{G}^*) = 2\mathrm{t}^{\sigma}_{\nu}(\mathrm{G}^*, \text{cons}) - 2\mathrm{t}^{\sigma}_{\nu}(\mathrm{G}^*, \text{source}) = 0$. Note that definition (116) of S^{σ}_{ν}, which is the one given in (Einstein 1916c, 1114, eq. 14), differs by a factor 2 from definition (20), which is the one given in (Einstein 1914c, 1075, eq. 76a).

143 In the discarded appendix to (Einstein 1916a), the covariance properties are also discussed first. Einstein never gets to energy-momentum conservation in that document (CPAE 6, Doc. 31).

The first term can be written as (see sec. 3.3, eq. (29))

$$\Delta \mathrm{G}^* = \left\{ 2g^{\alpha\nu}\frac{\partial \mathrm{G}^*}{\partial g^{\alpha\sigma}} + 2g_\alpha^{\beta\nu}\frac{\partial \mathrm{G}^*}{\partial g_\alpha^{\beta\sigma}} - g_\sigma^{\alpha\beta}\frac{\partial \mathrm{G}^*}{\partial g_\nu^{\alpha\beta}} \right\}\frac{\partial \Delta x^\sigma}{\partial x^\nu} + 2\frac{\partial \mathrm{G}^*}{\partial g_\alpha^{\mu\sigma}} g^{\mu\nu}\frac{\partial^2 \Delta x^\sigma}{\partial x^\nu \partial x^\alpha};$$

the second term as[144]

$$\sqrt{-g}\,\Delta\left(\frac{1}{\sqrt{-g}}\right)\mathrm{G}^* = \mathrm{G}^*\delta_\sigma^\nu\frac{\partial \Delta x^\sigma}{\partial x^\nu}.$$

Inserting these expressions into eq. (119), one finds

$$\sqrt{-g}\,\Delta\left(\frac{\mathrm{G}^*}{\sqrt{-g}}\right) = \left\{ 2g_\alpha^{\beta\nu}\frac{\partial \mathrm{G}^*}{\partial g_\alpha^{\beta\sigma}} + 2g^{\alpha\nu}\frac{\partial \mathrm{G}^*}{\partial g^{\alpha\sigma}} + \delta_\sigma^\nu \mathrm{G}^* - g_\sigma^{\alpha\beta}\frac{\partial \mathrm{G}^*}{\partial g_\nu^{\alpha\beta}} \right\}\frac{\partial \Delta x^\sigma}{\partial x^\nu} + 2\frac{\partial \mathrm{G}^*}{\partial g_\alpha^{\mu\sigma}} g^{\mu\nu}\frac{\partial^2 \Delta x^\sigma}{\partial x^\nu \partial x^\alpha}. \tag{120}$$

The expression in curly brackets is just S_σ^ν as defined in eq. (116). Eq. (120) can thus be written more compactly as

$$\sqrt{-g}\,\Delta\left(\frac{\mathrm{G}^*}{\sqrt{-g}}\right) = S_\sigma^\nu\frac{\partial \Delta x^\sigma}{\partial x^\nu} + 2\frac{\partial \mathrm{G}^*}{\partial g_\alpha^{\mu\sigma}} g^{\mu\nu}\frac{\partial^2 \Delta x^\sigma}{\partial x^\nu \partial x^\alpha} \tag{121}$$

(Einstein 1916c, 1114, eq. 13). Since G^* transforms as a scalar under arbitrary linear transformations,

$$S_\sigma^\nu = 0 \tag{122}$$

(ibid., eq. 15). The only contribution to the action comes from the second term on the right-hand side of eq. (121). Through partial integration this contribution can be rewritten as

$$\int 2\frac{\partial^2}{\partial x^\nu \partial x^\alpha}\left(\frac{\partial \mathrm{G}^*}{\partial g_\alpha^{\mu\sigma}} g^{\mu\nu}\right)\Delta x^\sigma d\tau$$

plus surface terms that will vanish. The invariance of $\int \mathrm{G}^* d\tau$ thus implies that

144 Using that $\Delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\Delta g^{\mu\nu}$ (eq. (39)), one can write

$$\Delta\left(\frac{1}{\sqrt{-g}}\right) = \frac{1}{g}\Delta\sqrt{-g} = \frac{1}{2\sqrt{-g}} g_{\mu\nu}\Delta g^{\mu\nu}.$$

Using that $\Delta g^{\mu\nu} = g^{\mu\alpha}\frac{\partial \Delta x^\nu}{\partial x^\alpha} + g^{\nu\alpha}\frac{\partial \Delta x^\mu}{\partial x^\alpha}$ (eq. (25) and Einstein 1916c, 1114, eq. 11), one finds:

$$\sqrt{-g}\,\Delta\left(\frac{1}{\sqrt{-g}}\right) = \frac{1}{2}\left(2g_{\mu\nu}g^{\mu\alpha}\frac{\partial \Delta x^\nu}{\partial x^\alpha}\right) = \delta_\nu^\alpha\frac{\partial \Delta x^\nu}{\partial x^\alpha},$$

from which the equation below follows.

$$B_\sigma = \frac{\partial^2}{\partial x^\nu \partial x^\alpha}\left(\frac{\partial \mathfrak{G}^*}{\partial g^{\mu\sigma}_\alpha} g^{\mu\nu}\right) = 0\,. \tag{123}$$

These are the contracted Bianchi identities. The general covariance of (the Lagrangian for) the Einstein field equations thus results in two identities—$S^\nu_\sigma = 0$ and $B_\sigma = 0$—that guarantee energy-momentum conservation (see eq. (117)).

For Einstein this was the central point of the paper. Writing to Ehrenfest, he summed up the paper as follows: "I have now given a Hamiltonian [read: variational] treatment of the essential points of general relativity as well, in order *to bring out the connection between relativity and the energy principle*" (our emphasis).[145] He told four other correspondents the same thing. Shortly after he had submitted the paper, he wrote to Besso: "You will soon receive a short paper of mine about the foundations of general relativity, in which it is shown how the requirement of relativity is connected with the energy principle. It is very amusing."[146] Similarly, he wrote to De Sitter a few days later: "Take a look at the page proofs [of (Einstein 1916c)] that I sent to Ehrenfest. There the connection between relativity postulate and energy law is brought out very clearly."[147] A little over a week later, he sent Lorentz an offprint of the paper describing it as "a short paper, in which I explained how in my opinion the relation of the conservation laws to the relativity postulate is to be understood."[148] He emphasized that the conservation laws are satisfied for any choice of the Lagrangian $\mathfrak{M}$ for matter, adding: "So the choice [of $\mathfrak{M}$] made by Hilbert appears to have no justification."[149] He made the same point in a letter to Weyl in which he once more reiterated the key point of (Einstein 1916c), namely that "[t]he connection between the requirement of general covariance and the conservation laws is also made clearer."[150]

Two years earlier Einstein had already made the connection between covariance and energy-momentum conservation in the context of the "Entwurf" theory (Einstein 1914c). He had shown that the conditions $B_\mu = 0$ and $S^\nu_\sigma = 0$ that in conjunction

145 "Ich habe nun das Prinzipielle an der allgemeinen Relativitätstheorie auch hamiltonisch dargestellt, um den Zusammenhang zwischen Relativität und Energieprinzip zu zeigen." Einstein to Paul Ehrenfest, 24 October 1916 (CPAE 8, Doc. 269).

146 "Du erhältst bald eine kleine Arbeit von mir über die Basis der allgemeinen Relativitätstheorie, in der gezeigt wird, wie die Rel-Forderung mit dem Energieprinzip zusammenhängt. Es ist sehr amusant." Einstein to Michele Besso, 31 October 1916 (CPAE 8, Doc. 270).

147 "Sehen Sie sich die Druckbogen an, die ich Ehrenfest geschickt habe. Es kommt dort der Zusammenhang zwischen Relat. Postulat und Energiesatz besonders klar heraus." Einstein to Willem de Sitter, 4 November 1916 (CPAE 8, Doc. 273).

148 "eine kleine Arbeit, in der ich dargestellt habe, wie nach meiner Ansicht die Beziehung der Erhaltungssätze zum Relativitätspostulat aufgefasst werden soll." Einstein to H. A. Lorentz, 13 November 1916 (CPAE 8, Doc. 276).

149 "Die von Hilbert getroffene Wahl erscheint daher durch nichts gerechtfertigt." Ibid.

150 "Auch wird der Zusammenhang zwischen allgemeiner Kovarianz-Forderung und Erhaltungssätzen deutlicher." Einstein to Hermann Weyl, 23 November 1916 (CPAE 8, Doc. 278). This is the same letter in which Einstein sharply criticizes Hilbert's adherence to the electromagnetic program (see note 139).

with the “Entwurf” field equations guarantee energy-momentum conservation also determine the class of “justified transformations” between “adapted coordinates” (see sec. 3.3). In the review article (Einstein 1916a), he had not connected the conditions guaranteeing energy-momentum conservation in unimodular coordinates to the corresponding covariance of the field equations. Instead, he had shown by direct calculation that these conditions are identically satisfied as long as unimodular coordinates are used (Einstein 1916a, sec. 17, eq. 55; cf. eq. (106) and notes 123 and 124). As he wrote to Ehrenfest: “In my earlier presentation [in Einstein 1916a] with $\sqrt{-g} = 1$, direct calculation establishes the identity that is here [in Einstein 1916c] presented as a consequence of the invariance [of the action].”[151] The variational treatment in arbitrary coordinates in (Einstein 1916c) thus fills two important gaps. The paper explicitly shows that energy-momentum conservation holds in arbitrary and not just in unimodular coordinates. More importantly, it establishes for the new theory what Einstein had already found for the old one, namely that there is an intimate connection between covariance and conservation laws.

It is no coincidence that the generalization of Einstein’s insight—the celebrated Noether theorems—was formulated only two years later. Energy-momentum conservation in general relativity was hotly debated in Göttingen following the abstruse treatment of the topic in (Hilbert 1915).[152] In the course of his first attempt to make sense of this part of Hilbert’s paper, Felix Klein (1917) claimed that energy-momentum conservation is an identity in Einstein’s theory. Klein claimed—or, to be more charitable to Klein, Einstein took him to claim—that eq. (113) holds as a direct consequence of the invariance of the action (107), *independently of the field equations*. In fact, only eq. (114) is an identity (see eq. (123)). And it is only in conjunction with the field equations (111), that this identity implies energy-momentum conservation as expressed in eq. (113). Einstein immediately set Klein straight on this score.[153] This was the start of a correspondence between the two men about energy-momentum conservation in general relativity.[154]

The debate quickly shifted from the status of the identities flowing from the general covariance of the action to the (related) issue of whether or not it was acceptable

151 “In meiner früheren Darstellung mit $\sqrt{-g} = 1$ wird die Identität direkt durch Ausrechnen konstatiert, welche hier als Folge der Invarianz dargestellt wird.” Einstein to Paul Ehrenfest, 7 November 1916 (CPAE 8, Doc. 275).

152 In the letter to Ehrenfest from which we already quoted in notes 127 and 133, Einstein vented his irritation with (Hilbert 1915): “I do not care for Hilbert’s presentation. It is […] unnecessarily complicated, not honest (= Gaussian) in its structure (creating the impression of being an *übermensch* by obfuscating one’s methods)” (“Hilberts Darstellung gefällt mir nicht. Sie ist […] unnötig kompliziert, nicht ehrlich (= Gaussisch) im Aufbau (Vorspiegelung des Übermenschen durch Verschleierung der Methoden).”). Our assessment of Hilbert’s paper follows (Renn and Stachel 1999). For a more positive assessment, see (Sauer 1999).

153 See the first paragraph of Einstein to Felix Klein, 13 March 1918 (CPAE 8, Docs. 480).

154 The most interesting letters are the first three, all written in March 1918: (1) the letter cited in note 153; (2) Felix Klein to Einstein, 20 March 1918 (CPAE 8, Doc. 487); (3) Einstein to Felix Klein, 24 March 1918 (CPAE 8, Doc. 492).

in a generally-covariant theory to have a non-generally-covariant gravitational energy-momentum tensor. Unaware that Lorentz (1916b, c) and Levi-Civita (1917) had already made the same proposal, Klein suggested to define the left-hand side of the gravitational field equations as the generally-covariant gravitational energy-momentum tensor. Like Lorentz and Levi-Civita before him, Klein even wrote a paper on this proposal, which Einstein convinced him not to publish.[155]

Einstein was virtually alone at this point in his defense of the pseudo-tensor. As he stated with his usual flair for high drama in the first sentence of (Einstein 1918d): "While the general theory of relativity has met with the approval of most physicists and mathematicians, almost all my colleagues object to my formulation of the energy-momentum law."[156] Einstein was unfazed by the opposition. Drawing heavily on (Einstein 1916c), (Einstein 1918d) provides a sustained defense of the views on energy-momentum conservation that had guided Einstein in finding and consolidating the "Entwurf" theory in 1913–1914 and that had guided him again in finding and consolidating its successor theory in 1915–1916. Subsequent developments would prove Einstein right. We now know that gravitational energy-momentum is represented by a pseudo-tensor and not by a tensor because gravitational energy-momentum cannot be localized.

Klein meanwhile continued to discuss the problem of energy-momentum conservation with other Göttingen mathematicians, notably with Carl Runge and Emmy Noether—one of the wrong sex, the other too old to be sent to the front. With Runge he undertook a systematic survey of the relevant literature. These efforts resulted in two important papers on the topic (Klein 1918a, 1918b). Noether's work on the problem resulted in her seminal paper on symmetries and conservation laws (Noether 1918).[157]

The importance of (Einstein 1916c) for our story is not so much its role in the run-up to Noether's theorems, but the evidence it provides for the continuity of Einstein's reliance on the variational formalism of (Einstein 1914c) in the transition from the "Entwurf" theory to general relativity. There was no abrupt break, no sudden switch from physical to mathematical strategy. Instead, the transition was brought about by changing one key element in the formalism encoding much of the physical knowledge that went into the "Entwurf" theory and then modifying other parts of the formalism (if necessary) to accommodate the new version of this one element. Einstein

155 Two drafts of this paper can be found in the Klein *Nachlass* in the Niedersächsische Staats- und Universitätsbibliothek in Göttingen (see CPAE 7, Doc. 9, note 5, for more details).

156 "Während die allgemeine Relativitätstheorie bei den meisten theoretischen Physikern und Mathematikern Zustimmung gefunden hat, erheben doch fast alle Fachgenossen gegen meine Formulierung des Impuls-Energiesatzes Einspruch" (Einstein 1918d, 448). This paper pulls together and amplifies earlier comments in (Einstein 1918a, sec. 6), written in response to Levi-Civita, and (Einstein 1918b), written in response to one of two short, little-known, and inconsequential excursions into general relativity by Erwin Schrödinger (1918). For discussion of the debate over energy-momentum conservation between Einstein and Levi-Civita, see (Cattani and De Maria 1993).

157 In broad outline this story can be found in (Rowe 1999). For a particularly illuminating analysis of Noether's theorems and their applications in physics, see (Brading 2002).

himself pinpointed this one element for us. It was the definition of the components of the gravitational field. Not all modifications necessitated by changing this definition were in place by the time he published the first of his four communications of November 1915 to the Prussian Academy (Einstein 1915a). Most of them were in place by the time he published the fourth (Einstein 1915d). This Einstein made clear in his systematic exposition of the theory in (Einstein 1916a). Even this paper, however, left at least one important question unanswered (viz., do the field equations guarantee energy-momentum conservation in arbitrary coordinates) and failed to transfer at least one important insight from the "Entwurf" theory to the new theory (viz., the relation between covariance and energy-momentum conservation). These issues were settled only with (Einstein 1916c), a paper that can be seen as the end of the consolidation phase of the theory, although one can argue that this phase was not brought to a conclusion until the publication of (Einstein 1918a, c, d).

10. HOW EINSTEIN REMEMBERED HE FOUND HIS FIELD EQUATIONS

In his papers of November 1915, Einstein introduced his new field equations by arguing that they were the natural choice given the central role of the Riemann tensor in differential geometry. The field equations are thus presented as a product of what we have called the mathematical strategy. The continuity with the "Entwurf" field equations, a product of the physical strategy, is lost in Einstein's presentation and the reader is left with the impression that Einstein abruptly switched from the physical to the mathematical strategy in the fall of 1915. This is exactly how Einstein himself came to remember the breakthrough of November 1915. The physics, he felt, had been nothing but a hindrance; he had been saved at the eleventh hour by the mathematics. In his later years Einstein routinely used this version of events to justify the purely mathematical approach in his work in unified field theory.

The way Einstein presented his new field equations in the first of his four papers of November 1915 (Einstein 1915a) is very different from the way we claim he found them. The paper opens with the retraction of the uniqueness argument of (Einstein 1914c) for the "Entwurf" field equations. After explaining what is wrong with this argument, Einstein writes in the third paragraph:

> For these reasons I completely lost confidence in the field equations I had constructed and looked for a way that would constrain the possibilities in a natural manner. I was thus led back to the demand of a more general covariance of the field equations, which I had abandoned with a heavy heart three years ago when I was collaborating with my friend Grossmann. In fact, back then we already came very close to the solution of the problem given below.[158]

The fourth paragraph announces a new theory in which all equations, including the field equations, are covariant under arbitrary unimodular transformations. Einstein does not explain, neither in this paragraph nor anywhere else in the paper, what made him forgo *general* covariance at this point. Our explanation is that the physical strategy pointed not to the generally-covariant Ricci tensor but to the November tensor, which only transforms as a tensor under unimodular transformations. We showed how changing the definition of the gravitational field set in motion a chain of reasoning that led from the "Entwurf" field equations to field equations based on the November tensor. Reading the passage quoted above, one would not have suspected such continuity. In fact, Einstein's revelations that he has "completely lost confidence" in the "Entwurf" field equations and that he had already come "very close to the [new] solution" three years earlier suggest a dramatic about-face. The fifth and final paragraph of Einstein's introduction confirms this impression and suggests an abrupt switch from the physical to the mathematical strategy:

> Hardly anybody who has truly understood the theory will be able to avoid coming under its spell. It is a real triumph of the method of the general differential calculus developed by Gauss, Riemann, Christoffel, Ricci, and Levi-Civita.[159]

This impression is further reinforced by the way in which the field equations are introduced in the paper. In sec. 2, on the construction of quantities transforming as tensors under unimodular transformations, Einstein shows how to extract the November tensor R_{ij} from the Ricci tensor $G_{ij} = R_{ij} + S_{ij}$ (Einstein 1915a, 782, eqs. (13), (13a), (13b)) just as he had done in the Zurich Notebook.[160] At the beginning of sec. 3, he then writes:

> After what has been said so far, it is natural to posit field equations of the form $R_{\mu\nu} = -\kappa T_{\mu\nu}$, because we already know that these equations are covariant under arbitrary transformations of determinant 1.[161]

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158 "Aus diesen Gründen verlor ich das Vertrauen zu den von mir aufgestellten Feldgleichungen vollständig und suchte nach einem Wege, der die Möglichkeiten in einer natürlichen Weise einschränkte. So gelangte ich zu der Forderung einer allgemeineren Kovarianz der Feldgleichungen zurück, von der ich vor drei Jahren, als ich zusammen mit meinem Freunde Grossmann arbeitete, nur mit schwerem Herzen abgegangen war. In der Tat waren wir damals der im nachfolgenden gegebenen Lösung des Problems bereits ganz nahe gekommen" (Einstein 1915a, p. 778). The Zurich Notebook shows that Einstein and Grossmann did indeed consider field equations based on the November tensor three years earlier (see sec. 2). Norton (2000, 150) inaccurately translates "gelangte … zurück" as "went back" rather than as "was led back." The difference is not unimportant. Norton's "went back" conveys discontinuity: Einstein abandoned one approach and adopted another, characterized by "the demand of a more general covariance." On this reading "a more general covariance" sounds odd. One would have expected "general covariance." Our "was led back" conveys continuity: staying the course Einstein ended up with "the demand of a more general covariance." On our reading "demand" sounds odd. One would have expected "property" or "feature" instead. Our reading, however, does fit with Einstein's remark quoted at the beginning of sec. 3 (see note 32): "The series of my papers on gravitation is a chain of erroneous paths, which nonetheless gradually brought me closer to my goal" (Einstein to H. A. Lorentz, 17 January 1916 [CPAE 8, Doc. 183])

159 "Dem Zauber dieser Theorie wird sich kaum jemand entziehen können, der sie wirklich erfaßt hat; sie bedeutet einen wahren Triumph der durch Gauss, Riemann, Christoffel, Ricci, und Levi-Civita begründeten Methode des allgemeinen Differentialkalküls" (Einstein 1915a, p. 779).

160 See eqs. (1)–(5) in sec. 2; G_{ij}, R_{ij}, and S_{ij} are defined in the equations following eq. (83).

161 "Nach dem bisher Gesagten liegt es nahe, die Feldgleichungen in der Form $R_{\mu\nu} = -\kappa T_{\mu\nu}$ anzusetzen, da wir bereits wissen, daß diese Gleichungen gegenüber beliebigen Transformationen von der Determinante 1 kovariant sind" (Einstein 1915a, p. 783).

It is only at this point that Einstein gives the Lagrangian formulation of these field equations (Einstein 1915a, 784, eq. (17)) and goes through the argument demonstrating that they are compatible with energy-momentum conservation (see sec. 6).[162]

Not surprisingly in light of the above, some of the best modern commentators on (Einstein 1915a) have concluded that its author had abruptly switched strategies in the fall of 1915. The clearest, most concise and most explicit version of this account can be found in (Norton 2000).[163] At the beginning of sec. 5, "Reversal at the Eleventh Hour," Norton gives the following summary of the developments of fall 1915:

> *… aware of the flaws in his "Entwurf" theory, Einstein decided he could only find the correct theory through the expressions naturally suggested by the mathematics. He proceeded rapidly to the completion of the theory and the greatest triumph of his life […]. Einstein now saw the magic in mathematics.* (Norton 2000, 148)

He elaborates:

> In effect, [Einstein's] new tactic [in the fall of 1915] was to reverse his decision of 1913. When the physical requirements appeared to contradict the formal mathematical requirements, he had then chosen in favour of the former. He now chose the latter and, writing down the mathematically natural equations, found himself rapidly propelled towards a theory that satisfied all the requirements and fulfilled his 'wildest dreams'[164] […] Einstein's reversal was his Moses that parted the waters and led him from bondage into the promised land of his general theory of relativity—and not a moment too soon. Had he delayed, the promised land might well have been Hilbert's.[165] Einstein [1933b, 289] recalled how he 'ruefully returned to the Riemann curvature'. He now saw just how directly the mathematical route had delivered the correct equations in 1913 and, by contrast, how treacherous was his passage if he used physical requirements as his principal compass (Norton 2000, 151–152)

There is an amusing pair of quotations from letters to Besso that can be used as evidence for 'Einstein's reversal' (cf. Norton 2000, 152). In March 1914, reporting results that seemed to solidify the "Entwurf" theory (see sec. 3), Einstein told Besso:

> The general theory of invariants only proved to be an obstacle. The direct route proved to be the only feasible one. The only thing that is incomprehensible is that I had to feel my way around for so long before I found *the obvious* [our emphasis].[166]

162 The same pattern can be found in the review article. Einstein (1916a, 803–804) introduces the field equations by connecting them to the Riemann tensor and then proceeds to discuss them using the variational formalism. In the letter to Ehrenfest, however, that (as we argued in sec. 8) formed the blueprint for the discussion of the gravitational field equations in the review article, Einstein writes down the Lagrangian right a way and does not say a word about the connection between the field equations and the Riemann tensor (Einstein to Paul Ehrenfest, 24 January 1916 or later [CPAE 8, Doc. 185]). Of course, he could simply have omitted that part because that was not what Ehrenfest had trouble with.

163 In a recent paper, Jeroen van Dongen (2004, sec. 2) fully endorses Norton's account. He is more careful in his dissertation (Van Dongen 2002).

164 See Einstein to Michele Besso, 10 December 1915 (CPAE 8, Doc. 162).

165 See Corry et al. 1997 for conclusive evidence of Einstein's priority (cf. note 65 above)

166 "Die allgemeine Invariantentheorie wirkte nur als Hemmnis. Der direkte Weg erwies sich als der einzig gangbare. Unbegreiflich ist nur, dass ich so lange tasten musste, bevor ich das Nächstliegende fand." Einstein to Michele Besso, ca. 10 March 1914 (CPAE 5, Doc. 514).

In December 1915, Einstein used the same term to tell Besso the exact opposite: "the obvious" ("das Nächstliegende") now refers to the mathematically rather than the physically obvious:

> This time *the obvious* was correct; however Grossmann and I believed that the conservation laws would not be satisfied and that Newton's law would not come out in first approximation [our emphasis].[167]

These twin quotations seem to provide strong, if anecdotal, evidence for Einstein changing horses in the fall of 1915.

Even in late 1915, however, as the last quotation illustrates, Einstein mentioned physical as well as mathematical considerations. The same thing he told Besso he told Sommerfeld and Hilbert too:

> It is easy, of course, to write down these generally-covariant field equations but difficult to see that they are a generalization of the Poisson equation and not easy to see that they satisfy the conservation laws.[168]
>
> The difficulty did not lie in finding generally-covariant equations for the $g_{\mu\nu}$; this is easily done with the help of the Riemannian tensor. Rather it was difficult to recognize that these equations formed a generalization of Newton's laws and indeed a simple and natural generalization.[169]

To be sure, these references to physical considerations fit with the alleged 'reversal' from physics to mathematics. First, we have to keep in mind that Einstein had ulterior motives in emphasizing that the mathematics was easy and that getting the physics straight was the hard part. He wanted to downplay the importance of Hilbert's work. Einstein felt that Hilbert had only worked on the theory's mathematical formalism and had not wrestled with the physical interpretation of the formalism the way he had. Second, Einstein's comments on the difficulty of the physical interpretation of the field equations still suggest that the decisive breakthrough occurred in the mathematics and that the physics then fell into place. We have argued that it was just the

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167 "Diesmal ist das Nächstliegende das Richtige gewesen; Grossmann und ich glaubten, dass die Erhaltungssätze nicht erfüllt seien, und das Newton'sche Gesetz in erster Näherung nicht herauskomme." Einstein to Michele Besso, 10 December 1915 (CPAE 8, Doc. 162).

168 "Es ist natürlich leicht, diese allgemein kovariante Gleichungen hinzusetzen, schwer aber, einzusehen, dass sie Verallgemeinerungen von Poissons Gleichungen sind, und nicht leicht, einzusehen, dass sie den Erhaltungssätzen Genüge leisten." Einstein to Arnold Sommerfeld, 28 November 1915 (CPAE 8, Doc. 153; see note 31 for a discussion of the context in which this letter was written). It is very telling that recovering the Poisson equation is presented as a problem that is harder than proving energy-momentum conservation. This is indeed the case—it requires overcoming the notion of coordinate restrictions and the prejudice about the form of the static metric—but one would never have guessed this from the November 1915 paper where the Poisson equation is recovered simply by applying the Hertz condition.

169 "Die Schwierigkeit bestand nicht darin allgemein kovariante Gleichungen für die $g_{\mu\nu}$ zu finden; denn dies gelingt leicht mit Hilfe des Riemann'schen Tensors. Sondern schwer war es, zu erkennen, dass diese Gleichungen eine Verallgemeinerung, und zwar eine einfache und natürliche Verallgemeinerung des Newton'schen Gesetzes bilden." Einstein to David Hilbert, 18 November 1915 (CPAE 8, Doc. 148).

other way around.

Why did Einstein nonetheless choose to present the new theory as a product of his mathematical strategy? Undoubtedly, part of the answer is that the key mathematical consideration pointing to the new field equations—the November tensor's pedigree in the Riemann tensor—is much simpler than the physical reasoning that had led Einstein to these equations in the first place. But even had the physical considerations been less forbidding, they would in all likelihood not have made for an effective and convincing argument for the new field equations. After all, Einstein had essentially drawn on these same considerations a year earlier for his fallacious argument for the uniqueness of the "Entwurf" field equations. That debacle was bound to come back and haunt a new argument along similar lines.

The math envy Einstein developed in the course of his work on general relativity may also have been a factor. As he confessed to Sommerfeld early on in his collaboration with Grossmann: "One thing for sure though is that I have never before in my life exerted myself even remotely as much and that I have been infused with great respect for mathematics, the subtler parts of which I until now, in my innocence, considered pure luxury. Compared to this problem, the original theory of relativity is child's play."[170] In 1917 he told Levi-Civita that "[i]t must be a pleasure to ride through these fields on the stallion of higher mathematics, while the likes of us have to muddle through on foot."[171]

We suspect, however, that Einstein's main reason for going with the mathematical argument was simply that he felt that this was by far the most persuasive argument in favor of the new field equations. Recall Einstein's satisfaction in October 1914 with the physical and mathematical lines of reasoning apparently converging on the "Entwurf" field equations, thereby finally rendering their covariance properties tractable (see our discussion at the end of sec. 4). If anything, the convergence of mathematical and physical lines of reasoning in late 1915 was more striking than it had been the year before. The concomitant clarification of the equations' covariance properties was accordingly more complete and more perspicuous. In the case of the "Entwurf" field equations, the clarification had taken the form of a complicated condition for non-autonomous transformations. In the case of the November tensor, the connection to the Riemann tensor immediately told Einstein that his new field equations were invariant under arbitrary unimodular transformations. Given how pleased Einstein had been with his much more modest result in 1914, this new result cannot have failed to impress him. As we saw at the end of sec. 4, Einstein got carried away

170 "Aber das eine ist sicher, dass ich mich im Leben noch nicht annäherend so geplag[t] habe, und dass ich grosse Hochachtung für die Mathematik eingeflösst bekommen habe, die ich bis jetzt in ihren subtileren Teilen in meiner Einfalt für puren Luxus ansah! Gegen dies Problem ist die ursprüngliche Relativitätstheorie eine Kinderei" (Einstein to Arnold Sommerfeld, 29 October 1912 [CPAE 5, Doc. 421]).

171 "Es muss hübsch sein, auf dem Gaul der eigentlichen Mathematik durch diese Gefilde zu reiten, während unsereiner sich zu Fuss durchhelfen muss" (Einstein to Tullio Levi-Civita, 2 August 1917 [CPAE 8, Doc. 368]).

by the earlier result, claiming that he had found definite field equations "in a completely formal manner, i.e., without direct use of our physical knowledge about gravity" (Einstein 1914c, 1076; cf. notes 78 and 79). The same happened in November 1915. In arguing for his new field equations, Einstein emphasized the covariance considerations to the exclusion of (at least) equally important considerations concerning energy-momentum conservation and the relation to Newtonian gravitational theory.

These physical considerations rapidly faded from memory. The way Einstein came to remember it, the general theory of relativity—the crowning achievement of his scientific career—was the result of a purely mathematical approach to physics. This distorted memory of how he had found general relativity served an important purpose in his subsequent career. Whenever the need arose to justify the speculative mathematical approach that never got him anywhere in his work on unified field theory, Einstein reminded his audience that he could boast of at least one impressive successful application of his preferred methodology.

The emblematic text documenting the later Einstein's extreme rationalist stance on scientific methodology is his Herbert Spencer lecture, held in Oxford on June 10, 1933.[172] This is where Einstein famously enthused that "[o]ur experience hitherto justifies us in believing that nature is the realization of the simplest conceivable mathematical ideas" (Einstein 1933a, 274[173]). Einstein routinely claimed that this was the lesson he had drawn from the way in which he had found general relativity. A few examples must suffice here. In a letter to Louis de Broglie the year before he died Einstein wrote that he arrived at the position expounded in his Spencer lecture

> through the experiences with the gravitational theory. The gravitational [field] equations could only be found on the basis of a purely formal principle (general covariance), i.e., on the basis of trust in the largest imaginable simplicity of the laws of nature.[174]

In his autobiographical notes of 1949, he similarly wrote that

> I have learned something else from the theory of gravitation: no collection of empirical facts however comprehensive can ever lead to the formulation of such complicated equations […] Equations of such complexity as are the equations of the gravitational field can be found only through the discovery of a logically simple mathematical condition that determines the equations completely or [at least] almost completely. Once one has those sufficiently strong formal conditions, one requires only little knowledge of facts for the setting up of a theory; in the case of the equations of gravitation it is the four-dimensionality and the symmetric tensor as expression for the structure of space which together with the invariance concerning the continuous transformation group, determine the equations almost completely.[175]

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172 For discussion, see, e.g., (Norton 2000), (Van Dongen 2002).

173 The German manuscript (EA 1 114) has: "Nach unserer bisherigen Erfahrung sind wir nämlich zu dem Vertrauen darin berechtigt, dass die Natur die Realisierung des mathematisch denkbar Einfachsten ist."

174 "durch die Erfahrungen bei der Gravitationstheorie. Die Gravitations-gleichungen waren nur auffindbar auf Grund eines rein formalen Prinzipes (allgemeine Kovarianz), d.h. auf Grund des Vertrauens auf die denkbar grösste logische Einfachheit der Naturgesetze" Einstein to Louis de Broglie, 8 February 1954. This letter is quoted and discussed in (Van Dongen 2002, 8)

Discussing this passage, Jeroen van Dongen (2002, 30) notes that it "reads like a unified field theory manifesto" and not as a "historically balanced account." As we have shown in this paper, the Einstein field equations were found not, as the later Einstein would have it, by extracting the mathematically simplest equations from the Riemann tensor, but by pursuing the analogy with Maxwell's equations for the electromagnetic field, making sure that they be compatible with Newtonian gravitational theory and energy-momentum conservation. Considerations of mathematical elegance played a role at various junctures but were always subordinate to physical considerations.

APPENDIX: THE TRANSITION FROM THE "ENTWURF" FIELD EQUATIONS TO THE EINSTEIN FIELD EQUATIONS SANITIZED

Drawing on calculations scattered throughout this paper and with the benefit of hindsight, we present a sanitized version of the path that took Einstein from the "Entwurf" field equations to the Einstein field equations. We start from the vacuum field equations in unimodular coordinates. In the form in which they were originally presented (Einstein and Grossmann 1913, 16-17, eqs. 15 and 18) the "Entwurf" equations look nothing like the Einstein field equations. In unimodular coordinates they can be written in a form that clearly brings out the relation to their successor.

Comparing Vacuum Field Equations. Both in the "Entwurf" theory and in general relativity, the vacuum field equations in unimodular coordinates can be derived from the action principle $\delta\int\sqrt{-g}L\,d\tau$. In both cases the Lagrangian is given by:

$$L = g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} \tag{A.1}$$

(see eq. (64) with $L \equiv \mathrm{L}$ for general relativity and eq. (36) with $L \equiv -H$ for the "Entwurf" theory). In general relativity the gravitational field is defined as (see eq. (54))

$$\Gamma^{\alpha}_{\beta\mu} \equiv -\left\{\begin{matrix}\alpha\\ \beta\mu\end{matrix}\right\} = -\frac{1}{2}g^{\alpha\rho}(g_{\rho\beta,\mu} + g_{\rho\mu,\beta} - g_{\beta\mu,\rho})\,; \tag{A.2}$$

in the "Entwurf" theory as (see eq. (53) with a minus sign)

$$\tilde{\Gamma}^{\alpha}_{\beta\mu} \equiv -\frac{1}{2}g^{\alpha\rho}g_{\rho\beta,\mu}\,. \tag{A.3}$$

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175 "Noch etwas anderes habe ich aus der Gravitationstheorie gelernt: Eine noch so umfangreiche Sammlung empirischer Fakten kann nicht zur Aufstellung so verwickelter Gleichungen führen […] Gleichungen von solcher Kompliziertheit wie die Gleichungen des Gravitationsfeldes können nur dadurch gefunden werden, dass eine logisch einfache mathematische Bedingung gefunden wird, welche die Gleichungen völlig oder nahezu determiniert. Hat man aber jene hinreichend starken formalen Bedingungen, so braucht man nur wenig Tatsachen-Wissen für die Aufstellung der Theorie; bei den Gravitationsgleichungen ist es die Vierdimensionalität und der symmetrische Tensor als Ausdruck für die Raumstruktur, welche zusammen mit der Invarianz der kontinuierlichen Transformationsgruppe die Gleichungen praktisch vollkommen determinieren" (Einstein 1949, 88–89).

To distinguish between corresponding quantities in the two theories, we shall write the "Entwurf" quantities with a tilde, as in eq. (A.3). Note that $\tilde{\Gamma}^{\alpha}_{\beta\mu}$ in eq. (A.3) is nothing but a truncated version of $\Gamma^{\alpha}_{\beta\mu}$ in eq. (A.2). Also note that the Lagrangian in eq. (A.1) is modelled on the Lagrangian for the free Maxwell field, $-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$.

The structural identity of the Lagrangians in the two theories does not carry over to the Euler-Lagrange equations. This is because of two complications. (1) The operations 'setting $\sqrt{-g} = 1$' and 'doing the variations' do not commute (see note 83). In the "Entwurf" theory we do the variations first. In general relativity we set $\sqrt{-g} = 1$ first. (2) The quantities $\Gamma^{\alpha}_{\beta\mu}$ are symmetric in their lower indices, whereas their counterparts, $\tilde{\Gamma}^{\alpha}_{\beta\mu}$, in the "Entwurf" theory are not.

In unimodular coordinates, the vacuum Einstein field equations can be written as (see note 89)[176]

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\nu\rho}\Gamma^{\lambda}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} = 0\,, \tag{A.4}$$

and the vacuum "Entwurf" field equations as (see eqs. (48) and (50)):

$$(g^{\alpha\beta}\tilde{\Gamma}^{\lambda}_{\mu\beta})_{,\alpha} - g^{\lambda\rho}\tilde{\Gamma}^{\alpha}_{\tau\mu}\tilde{\Gamma}^{\tau}_{\alpha\rho} + \frac{1}{2}\delta^{\lambda}_{\mu}g^{\rho\tau}\tilde{\Gamma}^{\alpha}_{\beta\rho}\tilde{\Gamma}^{\beta}_{\alpha\tau} = 0\,. \tag{A.5}$$

We can get these equations to resemble each other even more closely by defining

$$\Gamma^{\mu\nu}_{\alpha} \equiv g^{\mu\beta}\Gamma^{\nu}_{\alpha\beta} \tag{A.6}$$

in general relativity and the corresponding quantities

$$\tilde{\Gamma}^{\mu\nu}_{\alpha} \equiv g^{\nu\beta}\tilde{\Gamma}^{\mu}_{\alpha\beta} \tag{A.7}$$

in the "Entwurf" theory. Note that the order of the indices μ and ν is different in eqs. (A.6) and (A.7).[177] Inserting these quantities into eqs. (A.4) and (A.5), we find:

$$(\Gamma^{\lambda\alpha}_{\mu})_{,\alpha} - \Gamma^{\nu\lambda}_{\alpha}\Gamma^{\alpha}_{\nu\mu} = 0\,, \tag{A.8}$$

$$(\tilde{\Gamma}^{\lambda\alpha}_{\mu})_{,\alpha} - \tilde{\Gamma}^{\tau\lambda}_{\alpha}\tilde{\Gamma}^{\alpha}_{\tau\mu} + \frac{1}{2}\delta^{\lambda}_{\mu}\tilde{\Gamma}^{\alpha\tau}_{\beta}\tilde{\Gamma}^{\beta}_{\alpha\tau} = 0\,. \tag{A.9}$$

The first two terms in these two equation have the exact same form.

Expressed in unimodular coordinates and in terms of $\{\Gamma^{\mu}_{\alpha\beta}, \Gamma^{\alpha\beta}_{\mu}\}$ and $\{\tilde{\Gamma}^{\mu}_{\alpha\beta}, \tilde{\Gamma}^{\alpha\beta}_{\mu}\}$, respectively, the gravitational energy-momentum pseudo-tensors of the

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176 When the variation $\delta\int\sqrt{-g}L\,d\tau$ for L in eq. (A.1) is done *before* setting $\sqrt{-g} = 1$, one finds

$$\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} + \frac{1}{2}g_{\mu\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma} = 0$$

(see eq. (62) with $T_{\mu\nu} = 0$). Contracting this equation with $g^{\nu\lambda}$, one finds

$$(g^{\nu\lambda}\Gamma^{\alpha}_{\mu\nu})_{,\alpha} - g^{\nu\rho}\Gamma^{\lambda}_{\alpha\rho}\Gamma^{\alpha}_{\mu\nu} + \frac{1}{2}\delta^{\lambda}_{\nu}g^{\rho\sigma}\Gamma^{\alpha}_{\beta\rho}\Gamma^{\beta}_{\alpha\sigma} = 0$$

Not surprisingly, this last equation resembles eq. (A.5) in the "Entwurf" theory more closely than eq. (A.4), obtained when the variation is done *after* setting $\sqrt{-g} = 1$.

177 Expressed in terms of the new quantities $\Gamma^{\mu\nu}_{\alpha}$ and $\tilde{\Gamma}^{\mu\nu}_{\alpha}$, the Lagrangians for the two theories retain their structural identity: $L = g^{\mu\nu}\Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu} = \Gamma^{\alpha}_{\mu\beta}\Gamma^{\mu\beta}_{\alpha}$ and $\tilde{L} = g^{\mu\nu}\tilde{\Gamma}^{\alpha}_{\beta\mu}\tilde{\Gamma}^{\beta}_{\alpha\nu} = \tilde{\Gamma}^{\alpha}_{\beta\mu}\tilde{\Gamma}^{\beta\mu}_{\alpha}$.

two theories also take on the exact same form. With the help of eq. (A.6) the pseudo-tensor of general relativity in unimodular coordinates (see eq. (74)) can be written as

$$\kappa t_\sigma^\lambda = \frac{1}{2}\delta_\sigma^\lambda \Gamma_{\mu\beta}^\alpha \Gamma_\alpha^{\mu\beta} - \Gamma_{\mu\sigma}^\alpha \Gamma_\alpha^{\mu\lambda}. \tag{A.10}$$

With the help of eq. (A.7) the pseudo-tensor of the "Entwurf" theory in unimodular coordinates, with an overall minus sign because of the switch from H to $L = -H$ (see minus eq. (50) for $\sqrt{-g} = 1$) can be written as

$$\kappa \tilde{t}_\mu^\lambda = \frac{1}{2}\delta_\mu^\lambda \tilde{\Gamma}_{\beta\rho}^\alpha \tilde{\Gamma}_\alpha^{\beta\rho} - \tilde{\Gamma}_{\tau\mu}^\alpha \tilde{\Gamma}_\alpha^{\tau\lambda}. \tag{A.11}$$

Eqs. (A.10) and (A.11) have the exact same form. The first term in both expressions contains the trace of the pseudo-tensor:

$$\kappa t = \Gamma_{\mu\beta}^\alpha \Gamma_\alpha^{\mu\beta}, \qquad \kappa \tilde{t} = \tilde{\Gamma}_{\beta\rho}^\alpha \tilde{\Gamma}_\alpha^{\beta\rho}. \tag{A.12}$$

Using the expressions for the gravitational energy-momentum pseudo-tensors and their trace, we can write the field equations (A.8) and (A.9) in a form that brings out the physical interpretation of the various terms more clearly. Using eq. (A.12), we can rewrite eq. (A.10) as:

$$-\Gamma_\alpha^{\nu\lambda}\Gamma_{\nu\mu}^\alpha = \kappa t_\mu^\lambda - \frac{1}{2}\delta_\mu^\lambda \kappa t.$$

Substituting this equation into the field equations (A.8), we find[178]

$$(\Gamma_\mu^{\lambda\alpha})_{,\alpha} + \kappa t_\mu^\lambda - \frac{1}{2}\delta_\mu^\lambda \kappa t = 0. \tag{A.13}$$

Substituting eq. (A.11) into the field equations (A.9), we find

$$(\tilde{\Gamma}_\mu^{\lambda\alpha})_{,\alpha} + \kappa \tilde{t}_\mu^\lambda = 0. \tag{A.14}$$

The crucial difference between these last two equations is the trace term on the left-hand side of eq. (A.13).

From the "Entwurf" Field Equations to the Einstein Field Equations. Eqs. (A.13)–(A.14) suggest a short-cut for getting from the "Entwurf" field equations in unimodular coordinates to the Einstein field equations, first in unimodular coordinates and then in their generally-covariant form. Comparison of eq. (A.14) to eq. (A.13) shows that changing the definition of the gravitational field from $\tilde{\Gamma}_{\beta\mu}^\alpha$ in eq. (A.3) to $\Gamma_{\beta\mu}^\alpha$ in

178 If eq. (A.4) obtained by doing the variations *after* setting $\sqrt{-g} = 1$ are replaced by the equations

$$(g^{\nu\lambda}\Gamma_{\mu\nu}^\alpha)_{,\alpha} - g^{\nu\rho}\Gamma_{\alpha\rho}^\lambda \Gamma_{\mu\nu}^\alpha + \frac{1}{2}\delta_\nu^\lambda g^{\rho\sigma}\Gamma_{\beta\rho}^\alpha \Gamma_{\alpha\sigma}^\beta = 0,$$

obtained by doing the variations *before* setting $\sqrt{-g} = 1$ (see note 176), then eq. (A.13) gets replaced by $(\Gamma_\mu^{\lambda\alpha})_{,\alpha} + \kappa t_\mu^\lambda = 0$ which has the exact same form as eq. (A.14) in the "Entwurf" theory.

eq. (A.2) changes the way in which the gravitational energy-momentum pseudo-tensor occurs in the gravitational part of the field equations in unimodular coordinates. Since the energy-momentum of matter should enter the field equations in the same way as the energy-momentum of the gravitational field itself, this also affects the matter part of the field equations. In the presence of matter described by an energy-momentum tensor $T_{\mu\nu}$, the vacuum equations (A.14)—based on definition (A.3) of the gravitational field, Einstein's "fatal prejudice"—should be generalized to

$$(\tilde{\Gamma}_{\mu}^{\lambda\alpha})_{,\alpha} = -\kappa(\tilde{t}_{\mu}^{\lambda} + T_{\mu}^{\lambda}). \tag{A.15}$$

These are the "Entwurf" field equations in unimodular coordinates. By the same token, eq. (A.13)—based on definition (A.2) of the gravitational field, Einstein's "key to the solution"—should be generalized to:[179]

$$(\Gamma_{\mu}^{\lambda\alpha})_{,\alpha} = -\kappa\left([t_{\mu}^{\lambda} + T_{\mu}^{\lambda}] - \frac{1}{2}\delta_{\mu}^{\lambda}[t + T]\right). \tag{A.16}$$

These are the proper field equations for the successor theory to the "Entwurf" theory.

Eq. (A.15) guarantees energy-momentum conservation, $(\tilde{t}_{\mu}^{\lambda} + T_{\mu}^{\lambda})_{,\lambda} = 0$, in the "Entwurf" theory, if—in addition to $\sqrt{-g} = 1$—the condition

$$\tilde{B}_{\mu} \equiv (\tilde{\Gamma}_{\mu}^{\lambda\alpha})_{,\alpha\lambda} = 0 \tag{A.17}$$

holds (cf. eq. (51)).

Eq. (A.16) guarantees energy-momentum conservation, $(t_{\mu}^{\lambda} + T_{\mu}^{\lambda})_{,\lambda} = 0$, in the new theory if—in addition to $\sqrt{-g} = 1$—the condition

$$\left[(\Gamma_{\mu}^{\lambda\alpha})_{,\alpha} - \frac{1}{2}\delta_{\mu}^{\lambda}\kappa(t + T)\right]_{,\lambda} = 0 \tag{A.18}$$

holds. Contracting eq. (A.16), one finds that

$$(\Gamma_{\rho}^{\rho\alpha})_{,\alpha} = \kappa(t + T)$$

(see eq. (105)). Using this equation to eliminate T from eq. (A.18), one arrives at the condition $B_{\mu} = 0$ in the new theory

$$B_{\mu} \equiv \left[\Gamma_{\mu}^{\lambda\alpha} - \frac{1}{2}\delta_{\mu}^{\lambda}\Gamma_{\rho}^{\rho\alpha}\right]_{,\alpha\lambda} = 0 \tag{A.19}$$

(see eq. (106)). Eq. (A.19), it turns out, is an identity (see eq. (106) and notes 123–124). Eq. (A.17) in the "Entwurf" theory imposes a coordinate restriction over and above unimodularity. Eq. (A.19), its counterpart in the new theory, imposes no addi-

179 In his first November paper, Einstein (1915a) chose field equations in the presence of matter that set the left-hand side of eq. (A.13) equal to $-\kappa T_{\mu}^{\lambda}$. In the fourth November paper, Einstein (1915d) replaced the right-hand side by $-\kappa(T_{\mu}^{\lambda} - (1/2)\delta_{\mu}^{\lambda}T)$.

tional restriction.

The gravitational part of the field equations (A.16), i.e., the left-hand side of eq. (A.13), is nothing but an alternative expression for the November tensor $\Gamma^{\alpha}_{\mu\nu,\alpha} + \Gamma^{\alpha}_{\beta\mu}\Gamma^{\beta}_{\alpha\nu}$ (see eq. (68)), which itself is nothing but the Ricci tensor in unimodular coordinates (see eqs. (83)–(84)). It follows that the field equations (A.16) are (the mixed form of) the generally-covariant Einstein field equations,

$$R_{\mu\nu} = -\kappa\left(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}T\right), \tag{A.20}$$

in unimodular coordinates (where $R_{\mu\nu}$ is the Ricci tensor). Eq. (A.19) gives the contracted Bianchi identities in unimodular coordinates.

The generally-covariant form of the Einstein field equations can be derived from the action principle, $\delta\int\sqrt{-g}R d\tau = 0$, where R is the Riemann curvature scalar. All terms involving second-order derivatives of the metric can be eliminated from the action through partial integration. One then arrives at an action of the form

$$\int \mathrm{G}^* d\tau, \tag{A.21}$$

where

$$\mathrm{G}^* \equiv \sqrt{-g}g^{\mu\nu}\left[\begin{Bmatrix}\beta\\ \mu\alpha\end{Bmatrix}\begin{Bmatrix}\alpha\\ \nu\beta\end{Bmatrix} - \begin{Bmatrix}\alpha\\ \mu\nu\end{Bmatrix}\begin{Bmatrix}\alpha\\ \alpha\beta\end{Bmatrix}\right]. \tag{A.22}$$

In unimodular coordinates G^* reduces to

$$\mathrm{G}^*(\sqrt{-g} = 1) = g^{\mu\nu}\begin{Bmatrix}\beta\\ \mu\alpha\end{Bmatrix}\begin{Bmatrix}\alpha\\ \nu\beta\end{Bmatrix}. \tag{A.23}$$

This is just the Lagrangian given in eq. (A.1) with the gravitational field defined as minus the Christoffel symbols (see eq. (A.2)).

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