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Some Remarks on Prop. VIII Probl. II of Newton's Opticks Book I Part I

SOME REMARKS ON PROP. VIII PROBL. II OF NEWTON'S OPTICKS BOOK I PART I

by
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In Proposition VIII Problem II Newton is concerned with the following problem: Through a reflecting telescope of a given length and a given aperture radius a far remote object, e.g. a star, is seen with a certain degree of distinctness and brightness. The length and aperture radius of a second reflecting telescope is asked for through which the same object is seen with the same degree of distinctness and brightness, but in a different magnification.

Newton's own calculations have not come down to us. Robert Smith [1] seems to be the first to have published a solution of Newton's Proposition VIII Problem II. However, Smith's solution is not quite correct, and therefore a new solution is presented building soleley on the mathematics known to Newton and J. Lamberts's results [3] from his investigations of the brightness of luminous bodies.

The resolving power of a telescope is the measure of the distinctness with which an object is seen through the telescope. This resolving power is measured by the smallest distance between two points which can be recognized separately through the telescope. Since there is no chromatic aberration in concave mirrors, the resolving power of a reflecting telescope is merely limited by the spherical aberration of its concave mirror. To simplify matters, let us assume that the lens of the eye-piece is an ideal symmetrical lens without any spherical and chromatic aberration. In this case the diminuation of an object's distinctness is owes soleley to the spherical aberration of the concave mirror.

1. THE SPHERICAL ABERRATION OF A CONCAVE MIRROR

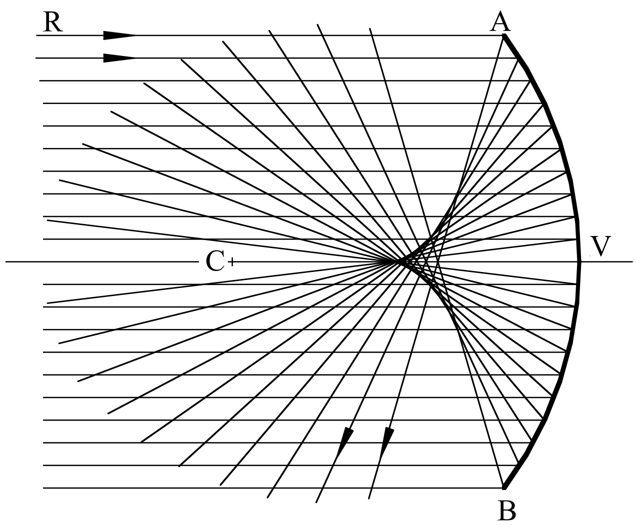


Fig. 1

In fig. 1 light falls parallel to the optical axis *CV* on the spherical concave mirror *AVB* with the vertex *V*. However, the reflected rays do not intersect the optical axis *CV* in a single point. Rays incident more distant from the optical axis before reflection have their point of intersection closer to the vertex *V*. This phenomenon is known as the spherical aberration of a spherical concave mirror.

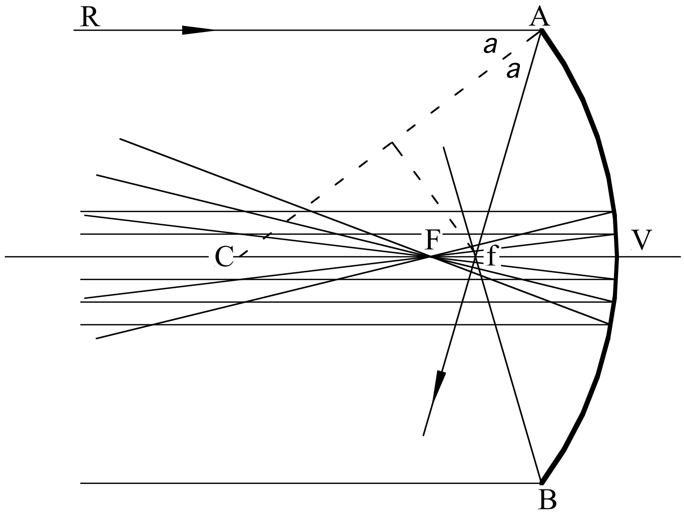


Fig. 2

In fig. 2 the radius *CA* of the spherical concave mirror *AVB* is perpendicular to the reflecting surface in the point *A*. Since the angle of incidence α is equal to the angle of reflection *CAf* and moreover this angle is equal to the angle *ACf*, the triangle *CAf* is isosceles. Let *r* denote the radius *CA* of the concave mirror *AVB*, then

$$\cos\alpha = \frac{\frac{1}{2}CA}{Cf} = \frac{r}{2Cf} \tag{1}$$

or

$$Cf = \frac{r}{2}\frac{1}{\cos\alpha}. \tag{2}$$

The distance *Vf* of the intersection point *f* from the vertex *V* is

$$Vf = CV - Cf = r - Cf. \tag{3}$$

By using the series

$$\frac{1}{\cos\alpha} = 1 + \frac{1}{2}\alpha^2 + \frac{5}{24}\alpha^4 + \frac{61}{720}\alpha^6 + \ldots \tag{4}$$

the relation (3) becomes

$$Vf = r - \frac{r}{2}\left(1 + \frac{1}{2}\alpha^2 + \ldots\right). \tag{5}$$

If the angle α is small, the distance Vf of the intersection point f of the ray RA is only slightly different from $\frac{r}{2}$ and if the ray RA is paraxial (i.e. $\alpha \approx 0$) the distance Vf equals $\frac{r}{2}$. The point F located in the middle between the vertex V und the center C of the concave mirror AVB is called the principal focus of the concave mirror. After their reflection the paraxial rays RA cross each other in this point F and therefore the distance CF is called the focal length of the concave mirror AVB. This focal length is

$$CF = FV = \frac{1}{2} CV. \tag{6}$$

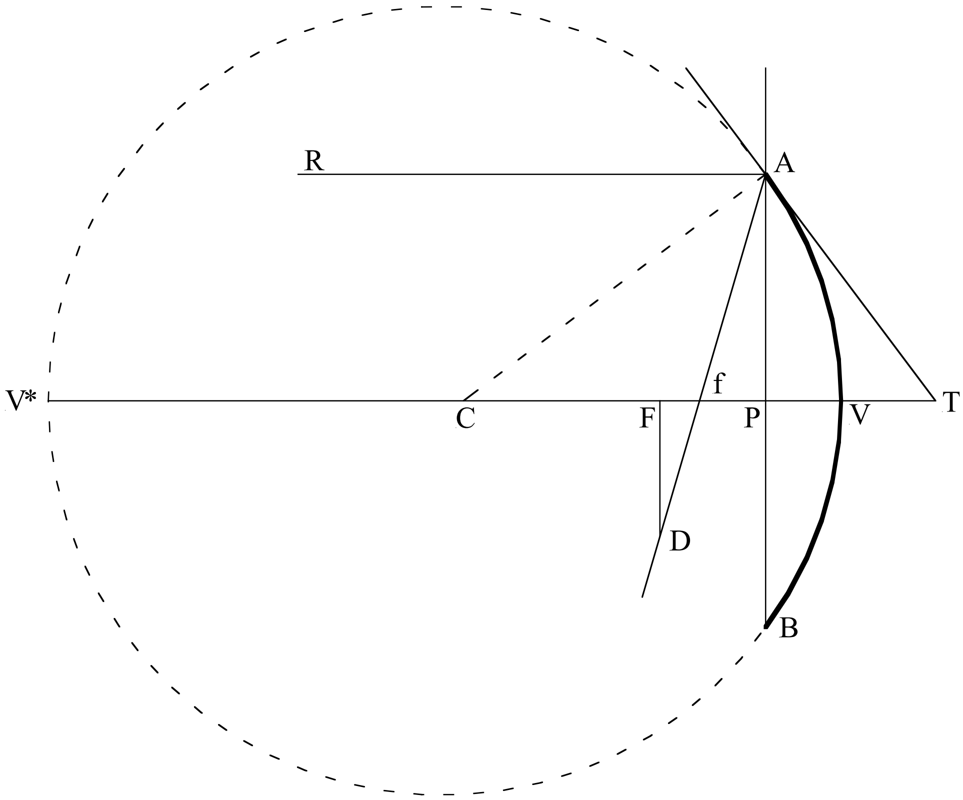


Fig. 3

In fig. 3 draw the tangent AT through the point A of the circle $AVBV^*$ and drop the perpendicular APB to the line V^*CV from the point A. Then the Pythagorean theorem and the altitude theorem of the triangle CAT yield the equations

$$CA^2 + AT^2 = CT^2 \tag{7}$$

$$CP \times PT = AP^2 \tag{8}$$

and the Pythagorean theorem of the triangle APT yields the equation

$$PT^2 + AP^2 = AT^2. \tag{9}$$

In view of $PT = CT - CP$ and $CV = CA$, these equations give rise to the proportion

$$CP : CV = CV : CT. \tag{10}$$

Moreover, this proportion yields the proportion

$$CP : CP - CV = CV : CV - CT. \tag{11}$$

And by using the relations $VP = CV - CP$ and $VT = CT - CV$ this proportion yields the proportion

$$CP : CV = VP : VT. \tag{12}$$

In the following let the rays RA always be paraxial, i.e. the rays RA incident on the concave mirror are parallel to the optical axis CV and very close to it. This assumption means that the aperture AB is sufficiently small for the approximations $VP \approx 0$ or $CP \approx CV$ to be valid, and because of (12) the approximation

$$VP \approx VT \tag{13}$$

is equally valid.

The triangle fAT is isosceles owing to the following angular relations in this triangle.

$$\sphericalangle fTA = \sphericalangle CTA = 90^0 - \sphericalangle ACT = 90^0 - \sphericalangle RAC \tag{14}$$

$$\sphericalangle fAT = 90^0 - \sphericalangle CAf = 90^0 - \sphericalangle RAC \tag{15}$$

Since the angles $\sphericalangle fTA$ and $\sphericalangle fAT$ are equal the equation

$$Af = fT \tag{16}$$

obtains. The triangle fCA is isosceles on account of

$$\sphericalangle fCA = \sphericalangle RAC \tag{17}$$

$$\sphericalangle CAf = \sphericalangle RAC. \tag{18}$$

Since the angles $\sphericalangle fCA$ and $\sphericalangle CAf$ are equal the equation

$$Cf = Af \tag{19}$$

is valid. The equality relation

$$Cf = Af = fT \tag{20}$$

is valid because of (16) und (19). The relation

$$Cf = \frac{1}{2} CT \tag{21}$$

follows from this equality relation because of $CT = Cf + fT$.

All the rays falling parallel to the optical axis CVT on the concave mirror AVB will upon reflection cut the optical axis CVT between the points f and F. The distance between these two points is called the longitudinal aberration fF of the concave mirror, i.e.

$$fF = Cf - CF. \tag{22}$$

With (6) and (21) the longitudinal aberration becomes

$$fF = \frac{1}{2}\left(CT - CV\right) = \frac{1}{2} VT. \tag{23}$$

Owing to (13) the longitudinal aberration is

$$fF = \frac{1}{2} VP. \tag{24}$$

To determine the dependence of this longitudinal aberration fF on the aperture radius AP and the radius CV of the concave mirror AVB we take into consideration the circle $AVBV^*$ with the diameter $VV^* = 2CV$. According to the theorem of Thales the triangle V^*AV is right-angled and therefore the altitude theorem applies, i.e.

$$AP^2 = V^*P \times VP \tag{25}$$

or

$$VP = \frac{AP^2}{V^*P}. \tag{26}$$

When the ray *RA* is paraxial, the length *VP* is very small and so the approximation $V^*P \approx V^*V = 2CV$ can be made use of. Then the relation (26) becomes

$$VP = \frac{1}{2}\frac{AP^2}{CV}. \tag{27}$$

Insertion of this relation into (24) yields the required expression of the longitudinal aberration *fF*, i.e.

$$fF = \frac{1}{4}\frac{AP^2}{CV}. \tag{28}$$

Upon reflection the whole lot of the rays incident on the concave mirror does not meet the optical axis in a single point, but the intersection points of the reflected rays are, as it were, smeared along the optical axis between *f* und *F*. For this reason the reflected rays do not meet precisely in the principal focus *F*, but are spread over a circle in the plane situated perpendicularly to the optical axis *CVT* in the principal focus *F*. This circle is known as the aberration circle in the principal focus *F* and his radius *FD* is known as the lateral aberration in the principal focus *F*. This lateral aberration *FD* is

$$FD = \frac{AP \times fF}{fP} \tag{29}$$

on account of the proportion

$$FD : fF = AP : fP. \tag{30}$$

Because of (28) and (29) the lateral aberration of the paraxial rays is

$$FD = \frac{1}{4}\frac{AP^3}{CV \times fP}. \tag{31}$$

To determine the dependence of this lateral aberration on the aperture radius *AP* and the radius of the curvature *CV* of the concave mirror *AVB* we use the following relation resulting from fig. 3

$$fP = CV - VP - fF - CF. \tag{32}$$

Insertion of (6) and (24) into this relation yields

$$fP = \frac{1}{2}CV - \frac{3}{2}VP. \tag{33}$$

Since in case of paraxial rays the length *VP* is very small the expression (33) becomes

$$fP = \frac{1}{2}CV. \tag{34}$$

Finally, by means of this expression the relation (31) yields the required relation

$$FD = \frac{1}{2}\frac{AP^3}{CV^2}. \tag{35}$$

The lateral aberration FD in the principal focus F is directly as the cube of the aperture radius AP of the concave mirror and inversely as the square of its curvature radius CV.

2. THE DISTINCTNESS OF A REFLECTING TELESCOPE

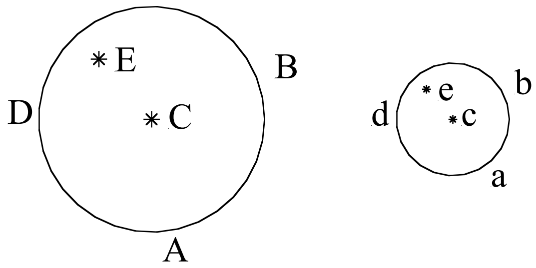
Through two different telescopes one and the same object is seen with the same distinctness when the same details are distinguishable. So allowing for spherical aberration, how closely may two details be located without merging?

The overly short answer provided by Robert Smith in his book [1, p. 139 § 342] was changed and more fully formulated by Abraham G. Kästner [2] in his German translation and revision of Smith's book. Kästner explains in great detail that the magnitude of the aberration circle of a lens in its focus is the measure of the distinctness in which a viewed object is seen through the lens. Kästner's considerations which only deal with the aberration of a lens can easily be applied to the concave mirrors of the reflecting telescopes. For this reason we will first present at some length the argument contained in Kästner's book [2, p. 142 - 144]:

> „What I will offer to bear out this proposition I do not wish to call a fully consistent proof, but hopefully will suffice for credibility to those who do not flatly deny the common doctrine of vision.
>
> For distinct vision it is required that every point of the object has just one point on the bottom of the eye as its image and that no other light falls on this point. This is corroborated by the common doctrine of sensations. For if a site of the eye is simultaneously affected by the light of other points, the soul will perceive simultaneously something from all these other points and in the process will not discriminate between different points because due to experience discrimination is only among points whose images fall on different parts of the eye. Therefore the sensation through an image containing light from different points of the object will consists of different sensations of different parts of the object, but these different sensations are not distinguished, i.e. the soul will have an indistinct sensation because according to the general laws of sensation and due to the limits which are set to the powers of the soul an indistinct sensation is composed of a multitude of sensations which arise simultaneously and cannot be distinguished by the soul. In a word, if many things simultaneously excite one single place of a sense organ, then the sensation excited thereby becomes blurred because this sensation is to represent something of all these things, but cannot fully distinguish one thing from another.
>
> I do not assert herby that the soul will not at all discriminate between parts of the object which send light to one and the same site of the eye. For from these very parts light can also fall on other locations of the eye and owing this the soul will discriminate among them. The point is, however, that the soul does not discriminate among them by light falling on a single site.
>
> Hence, if the rays coming from a single point of the object are spread out on the bottom of the eye across the circle ABD with the center C as a result of the refraction which the rays suffer when passing through the glasses, then this center C would be the distinct image of that point of the object and would contain all the rays coming from that point into the eye unless this would be changed by the refraction of the glasses: Because there is no reason to assume here that the glasses

would deflect the rays more to the one side than to the other side of the point toward which they would refract the rays if there were no spherical aberration and chromatic aberration.

Hence, the rays coming from another point of the object are, as a consequence of refraction by the lenses, spread out across another equally large circle (for there once again no reason to assume different magnitudes) whose center *E* be inside the circle *ABD*. It is evident that light coming from this second luminous point strikes *C* because light coming from that point spreads out across the whole circle having the center *E* and a semidiameter equal to the semidiameter of the circle *ABD*.



Therefore *C* receives light from so many points of the object as points *E* are inside the area *ABD*. Since presumably *C* receives as much light from some point as it receives from another point, the indistinctness is as the quantity of the points *E* being inside *ABD*.

If now *abd* is another aberration circle in the same eye or in an eye of the same constitution, but caused by other glasses, and if *c* is its center and *e* is the center of the aberration circle of the rays coming from another point of the object, then the quantity of the extraneous rays falling on *C* will be to the quantity of the extraneous rays falling on *c* as the quantity of points *E* contained in *ABD* is to the quantity of points *e* contained in *abd*. Since in both cases the constitution of the eye is the same and these circles are not caused by a defect of the eye, but by refraction taking place in the glasses, the points *C*, *E*, *c*, *e* would be perfectly distinct images of certain points of the object stopping short of the blurring caused by the refraction of the rays. But if the object remains the same and is seen distinctly through glasses by the same eye, then its image will always remain similar to itself, the magnitude of the image changing only when other glasses or differently composed glasses are assumed, i.e. the points *C*, *E* belonging to certain glasses will have the same position to each other as the points *c*, *e* belonging to other glasses. For example, the object would appear smaller through other glasses, i.e. its image in the eye would be smaller and yet distinct. If *C* is the image of a certain point of the object seen through the first glasses and *c* the image seen through the other ones and and this being also true of *E* and *e*, then the point *e* will have the same position to *c* as *E* has to *C* but with *E* lying closer to *C*. Innumerable points *e* lying around *c* will yield an image of the object which is similar to the image yielded by innumerable points *E* and *C,* being but smaller. Hence, the quantity of points inside the circle *ABD* will be to the quantity of points *e* inside the circle *abd* as are the areas of both circle to one another. These quantities, however, are as the indistinctnesses."

Now let us apply Kästner's considerations to the Newtonian-type reflecting telescop. In fig. 4 the concave mirror *AB*, the ocular lens *HH** and the eye or the pupil *PP** of the observer have been arranged in a single line presenting the optical axis (ignoring optically irrelevant complications through the prism).

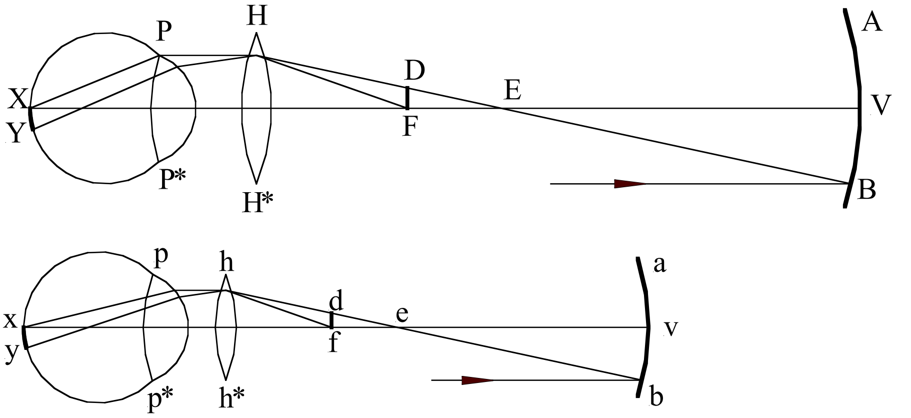


Fig. 4

The point F is the focus both of the concave mirror AB and of the ocular lens HH^*. When a light ray coming from this common focus F falls on the lens HH^*, after its passage through this lens it will continue parallel to the optical axis, pass through the pupil PP^* and meet the retina at point X. Owing to the spherical aberration of mirror AB, the outermost light ray falling parallel to the optical axis on concave mirror AB in B will cut the optical axis not in F, but in E. After its passage through the lens HH^* it will not run parallel to the optical axis, but slightly inclined toward the optical axis. And so it will meet the retina not in X, but in Y. Hence XY is the image of the aberration circle DF on the observer‘s retina.

Let a second Newtonian reflecting telescop similar to the former one be denoted by small letters. What has been said about the former also applies to the latter. According to Smith and Kästner, these two telescopes, then, yield equally distinct views whenever the images XY, xy of the aberration circles DF, df on the retina are equal. Therefore the relation of equality obtains:

$$XY = xy. \tag{36}$$

According to fig. 5 the magnitude XY of the image of the lateral aberration FD on the retina results from the following proportion

$$XY : \rho_{Auge} = FD : KO \tag{37}$$

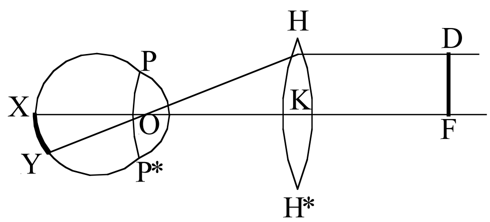


Fig. 5

where $\rho_{Auge} = XO$ is the diameter of the observer’s eye and $OK = FK$ the focal length of the eye piece.

For the telescope denoted by small letters the similar proportion obtains:

$$xy : \rho_{Auge} = fd : ko. \tag{38}$$

Insertion of the expressions of *XY*, *xy* resulting from these both proportions into the equation (36) yields

$$\frac{FD}{FK} = \frac{fd}{fk}. \tag{39}$$

By using the relation (35) the equation

$$\frac{AP^3}{ap^3} = \frac{CV^2 \times FK}{cv^2 \times fk} \tag{40}$$

or

$$\frac{FK}{fk} = \frac{AP^3 \times cv^2}{ap^3 \times CV^2}. \tag{41}$$

is obtained. This relation implies that one and the same object is seen equally distinct through two similar Newtonian reflecting telescopes when the cubes of the aperture diameters of their large mirrors are as the products of the squares of the focal distances of the large mirrors and the focal distances of the eye-pieces, or when the focal distances of the eye-pieces are as the cubes of the aperture diameters of the large mirrors divided by the squares of their focal distances.

3. The magnification of a Newtonian reflecting telescope

The visual angle of an object (i.e. the angular size or apparent size) is the angle φ which brackets the object when seen. The linear magnification of a telescope is defined as the quotient of the size of image on the retina when the object is seen through the telescope and

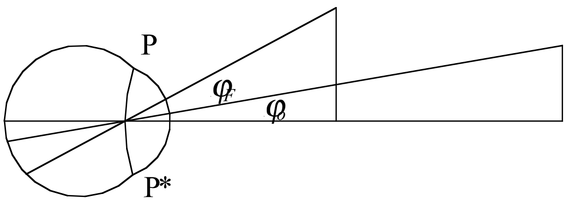


Fig. 6

when seen by the unaided eye. Therefore, when the object is seen in an angle φ_F with the telescope and in an angle φ_o without the telescope, the magnification is

$$M = \frac{\varphi_F}{\varphi_o}. \tag{42}$$

Because of the series

$$\alpha = \tan\alpha - \frac{1}{3}\tan^3\alpha + \frac{1}{5}\tan^5\alpha - ... \tag{43}$$

the angle α can be substituted by its tangens when α is very small. Then the magnification is

$$M = \frac{\tan\varphi_F}{\tan\varphi_o}. \tag{44}$$

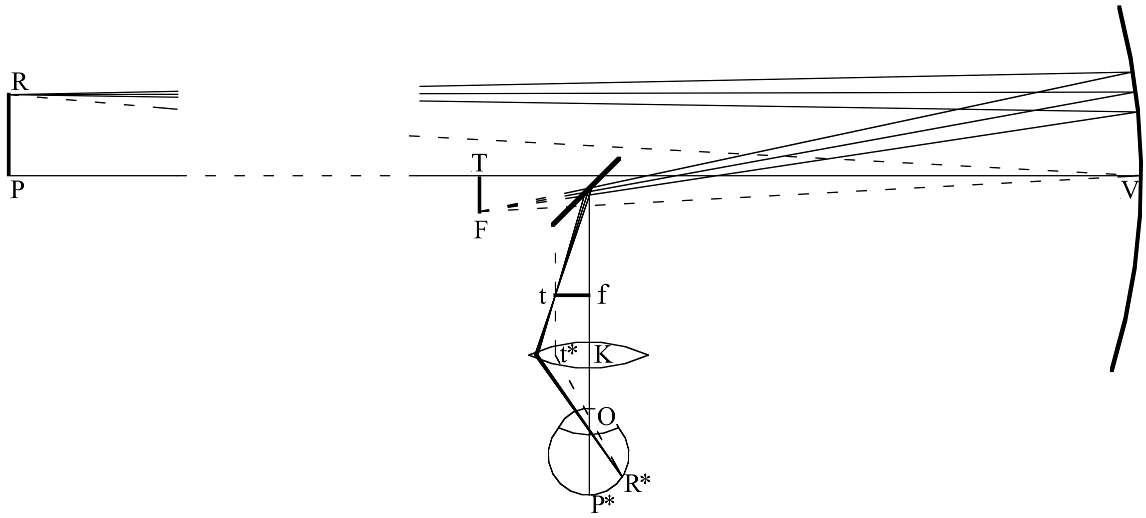


Fig. 7

In fig. 7 an object PR far away is viewed through a Newtonian reflecting telescope, the observer sees the image $TF = tf = t^*K$ at an angle $\varphi_F = \sphericalangle\, t^*OK \;= \sphericalangle R^*OP^*$, and without a telescope at an angle $\varphi_o = \sphericalangle\, RVP = \sphericalangle\, FVT$. Hence, the magnification (42) is

$$M = \frac{\sphericalangle\, t^*OK}{\sphericalangle\, RVP}. \tag{45}$$

Since the object PR is far away, the angles are very small. Therefore the angles can be approximated by their tangens, i.e. $\sphericalangle\, t^*OK \;= \frac{t^*K}{KO}$ and $\sphericalangle\, RVP = \sphericalangle\, FVT = \frac{TF}{TV} = \frac{tf}{TV}$. Because of $t^*K \;= tf$ the magnification (44) will be

$$M = \frac{TV}{KO}. \tag{46}$$

For the sake of simplicity a thin symmetrical ocular lens t^*K with the focus f is assumed. Because of $KO = fK$ magnification M of the Newtonian reflecting telescope will be the quotient of the focal distance TV of the concave mirror and the focal distance KO of the ocular.

4. THE CONDITION FOR THE EQUAL BRIGHTNESS

The apparent brightness of an observed object is that of its image on the observer's retina. The quantity of light coming from the object and passing through the observer's eye determines the creation of the image on the retina. The felt brightness of the image correlates positively with the quantity of light. Accordingly, the quotient given by the quantity of light falling on the area divided by the magnitude of this area provides a measure of this area's brightness, i.e. the density of the light on this area.

Using two different reflecting telescopes, a particular object elicites the same brightness when the images of the object on the retina are equally bright, or in other words, in both cases the light density of the images on the retina of the observer are equal.

Since the determination of equality of brightness of the same object seen through two different telescopes is a non-trivial problem, it is most unfortunate that Newton's own calculations for solving this problem are lost. Robert Smith [1, p. 141 § 348, 349] believes this problem is an easily solved one. He writes:

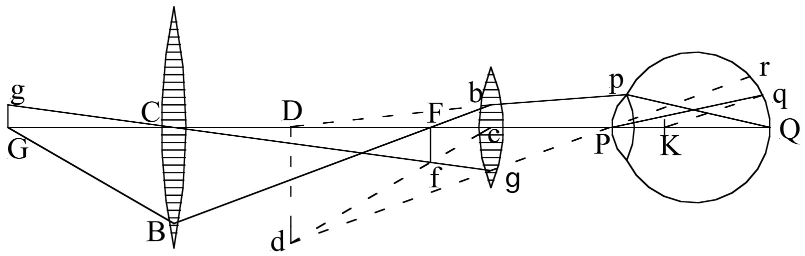
> In all sorts of telescopes and double microscopes the apparent brightness of a given object is as the square of their linear apertures directly and as the square of their linear amplifications inversely.
>
> For if the squares of the linear amplifications, that is if the areas of the pictures upon the retina were the same, their brightness would be as the quantities of light coming through the areas of the apertures, that is as the squares of the linear apertures; and if the apertures or quantities of light were the same, the brightness of the pictures would be as their areas inversely or as the squares of the linear amplifications inversely. Therefore when neither the apertures nor the amplifications are the same, the brigtness is as the square of the linear apertures directly, and as the square of the linear amplifications inversely.
>
> … Hence in refracting and reflecting telescopes a given object appears equally bright, when their linear apertures are as their linear amplifications, …

No further physical explanation is given by R. Smith in his book. And no amendment to his subject is given by A.G. Kästner [2] in his German translation and revision of Robert Smith's book. On the page 185 Kästner touches on the problem of how „to compare the brightness of two different telescopes“ but says only:

> „The brightness is as the quantity of light rays divided by the magnitude of the picture and the quantity of the light rays falling on both oculars is as the areas.“

These arguments are not really conclusive because the felt brightness of a seen object depends mainly on the intensity of sensation arising from the very image on the retina of the viewed object. Clearly, a change in a telescope's aperture immediately affects the brightness of the whole image produced by the telescope, but as a rule there is a concomitant change in the telescope's magnification, and this renders the given arguments insufficient to determine the brightness of the viewed object or of its image on the retina. And this is why the brightness of the image caused by the object has to be calculated directly. Johannes Heinrich Lambert [3] was the first to solve this problem for a Keplerian telescope. In Pars IV Cap. I p. 362 – 364 § 804 – 806 he treats the following problem [4, p. 274/5]:

§ 804. Let *PQ* be an eye, *GPQ* its axis, interposed are two lenses *bc*, *BC*, object *Gg* is viewed, and its apparent brightness is sought. Let *DP* be the distance, at which the eye sees the object distinctly, *Pp* the semidiameter of the aperture of th pupil, *Ff* the image of the object, *GBFbp* is the path of the rays from point *G*, which is on the axis, incident at the extremity of the pupil; *CB* will have to be the aperture of th objective lens and *bc* the aperture of th ocular lens, if all rays that are refracted in this way that can enter the pupil, do enter it; so even if both be larger, no more rays traverse the pupil. It is otherwise if one of the other aperture were to be smaller, then *Pp* will not be the entire pupil aperture, but only that part which rays fill heading to the eye from point *G*. Any aperture made smaller in any given case will be easy to determine from the principles of dioptrics. To us, this ray *GBFbp* will be the extreme of those which reach the retina from point *G*, which is on the axis.



§ 805. So that the central brightness can be defined, let *Gg* be the semidiameter of an element of space infinitely small, by drawing *gCf*, *Ff* will be its image; ray *gf* is incident at *γ*, where refracted it proceeds to *P*, where again refracted it is incident at point of the retina, *q*. By extending line *γP* to *r*, then approximately $qQ = \frac{2}{3}Qr$, or making $QK = \frac{2}{3}QP$ and drawing *Kq* parallel to *Pr*, so that *Qq* is the image of object *Gg* depicted on the retina. Finally, dropping a normal *Dd* and extending *Pγ* to *d*, *d* will be the point to where line *cf* is extended.

§ 806. Omitting again the quantity of light reflected and dispersed by the eye ad the lenses, we assume all the light from space *Gg*, incident on aperture *CB*, to be incident also on its image *Qq*. Whence, the brightness of its image will be given, if the quantity is divided by the space *Qq*. Calling this quantity = *q*, then

$$q = \frac{\pi\pi \cdot Gg^2 \cdot CB^2}{GB^2}.$$

But the area of space *Qq* is $= \pi \cdot Qq^2$, therefore taking the brightness at $Q = \eta$, then

$$\eta = \frac{\pi \cdot Gg^2 \cdot CB^2}{GB^2 \cdot Qq^2}.$$

The examination of J.H. Lambert's calculation shows that he obtained his results by using the infinitesimal calculus. However, Newton could have obtained similar results by means of his fluxion calculus. Lambert's results[1] can be applied straightforwardly to the Newtonian reflecting telescope because it equals a Keplerian telescope in that its mirror plays the part which the objectiv-glass has in the Keplerian telescope. Hence, the equations describing the

[1] Some readers perhaps think that there are some missprints in Lambert's formulas of the quantity q of light incident on the retina or the corresponding brightness η because these formulas do not explicitly contain the luminous intensity of the viewed star. This alleged mistake is a consequence from Lambert's standardization. See e.g. § 111 in [3].

optical properties of a Keplerian telescope also describe the optical properties of a Newtonian reflecting telescope. The optical quantities of the Keplerian telescope are equivalent to those of the Newtonian reflecting telescope.

Now let us consider two reflecting telescops of different magnification. In fig. 8 we denote the first by the capitel letters G, B, Q etc., and the second by the corresponding small letters g, b, q etc. Let the points F, f be the common focal point of the concave mirror VB and

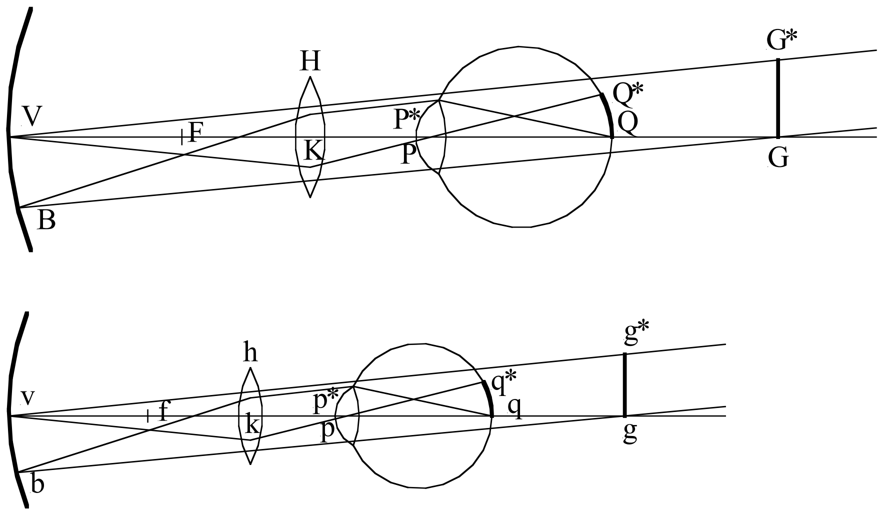


Fig. 8

the ocular lens H or of the concave mirror vb and the ocular lens h respectively. We now consider the case that through both telescopes a particular star, i.e. $GG^* = gg^*$, is seen with the same brightness. Therefore, $\mathrm{H} = \eta$ or

$$\frac{\pi\, GG^{*2} \times VB^2}{GB^2 \times QQ^{*2}} = \frac{\pi\, gg^{*2} \times vb^2}{gb^2 \times qq^{*2}} \tag{47}$$

is valid. Because of the great distance of the observed star the approximation $GB = gb$ is admissible. This approximation yields the equation

$$\frac{VB^2}{vb^2} = \frac{QQ^{*2}}{qq^{*2}}. \tag{48}$$

Let Φ_F, φ_F be the visual angles under which you see the viewed star with the two telescopes. Let φ_o be the visual angle without telescope. Then the two relation

$$QQ^* = \rho_{eye}\Phi_F \tag{49}$$

$$qq^* = \rho_{eye}\varphi_F \tag{50}$$

are true (ρ_{eye} is the diameter of the eye). These relations yield

$$\frac{M}{m} = \frac{QQ^*}{qq^*}, \tag{51}$$

with the linear magnifications $M = \dfrac{\Phi_F}{\varphi_o}$ and $m = \dfrac{\varphi_F}{\varphi_o}$, so that finally the equation (48) becomes

$$\frac{VB^2}{vb^2} = \frac{M^2}{m^2} \tag{52}$$

or

$$\frac{VB}{vb} = \frac{M}{m}. \tag{53}$$

Because of (46) the magnifications are $M = \dfrac{VF}{FK}$ and $m = \dfrac{vf}{fk}$. Insertion of these expressions into (53) yields

$$\frac{FK}{fk} = \frac{VF}{VB}\frac{vb}{vf}. \tag{54}$$

The relation (41) becomes

$$\frac{FK}{fk} = \frac{VB^3}{VF^2}\frac{vf^2}{vb^3}. \tag{55}$$

Finally, by equating (54) and (55) the relation

$$\frac{VB}{vb} = \left(\sqrt[4]{\frac{VF}{vf}}\right)^3. \tag{56}$$

is obtained. Since the lengths L, l of the Newtonian reflecting telescopes are approximately equal the focal distances VF, vf of their concave mirrors, the relation (56) yields Newton's result

$$\frac{VB}{vb} = \left(\sqrt[4]{\frac{L}{l}}\right)^3. \tag{57}$$

So the aperture diameters of reflecting telescopes through which the same object is seen with the same brightness and the same distinctness are as the cubes of the fourth roots of the lengths of the reflecting telescopes. Furthermore, the equation (53) shows that the aperture diameters are as the magnifications. All of this is encompassed by Newton's statement „If the Instrument be made longer or shorter, the aperture must be in proportion as the Cube of the square-square Root of the length, and the magnifying as the aperture."

5. On Robert Smith's solution

It is worth mentioning that Robert Smith [1, p. 141 § 348] started from an unproven physical statement when he attempted to derive Newton's proposition on when through different reflecting telescopes the same object is seen with the same distinctness and the same brightness.

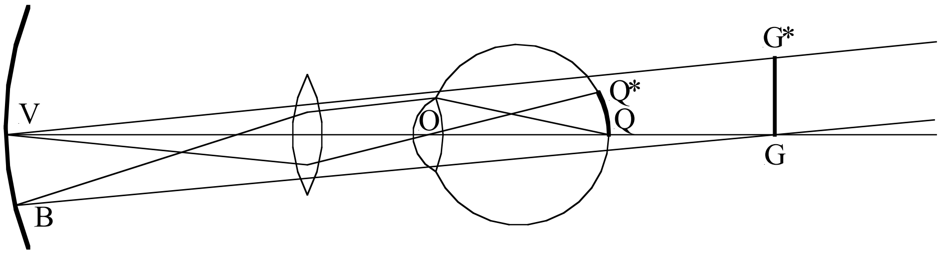


Fig. 9

Let φ_F denote the angle Q^*OQ under which the far distant object GG^* is seen with the telescope and let φ_o denote the angle G^*VG under which the object GG^* is seen without telescope. The following geometrical relations are valid (ρ_{eye} is the eye diameter):

$$QQ^* = \rho_{eye} \tan\varphi_F \tag{58}$$

$$GG^* = GV \tan\varphi_o . \tag{59}$$

According to Lambert the brightness η of the distant object GG^* is

$$\eta = \frac{\pi\, GV^2 \tan^2\varphi_o\, VB^2}{GB^2\, \rho_{eye}^2 \tan^2\varphi_F} . \tag{60}$$

Let δ be the angle BGV under which the aperture radius VB of the telescope is seen from the object GG^*. Because of

$$\cos\delta = \frac{GV}{GB} \tag{61}$$

the expression of the brightness η can be written as

$$\eta = \frac{\pi \cos^2\delta\ \tan^2\varphi_o\, VB^2}{\rho_{eye}^2 \tan^2\varphi_F} . \tag{62}$$

If the viewed object is very distant, δ will be very small, i.e. $\cos\delta = 1$. So

$$\eta = \frac{\pi\, VB^2}{\rho_{eye}^2} \frac{\tan^2\varphi_o}{\tan^2\varphi_F} . \tag{63}$$

The angles φ_F and φ_o may be assumed as very small. Then the magnification is

$$M = \frac{\varphi_F}{\varphi_o} = \frac{\tan\varphi_F}{\tan\varphi_o} . \tag{64}$$

Finally, the expression of the brightness η, with which the distant object GG^* is seen, becomes

$$\eta = \frac{\pi}{\rho_{eye}^2} \frac{VB^2}{M^2} . \tag{65}$$

Hence, the brightness of the observed object is directly proportional to the square of the aperture diameter *VB* and inversely proportional to the square of the linear magnification *M*. Robert Smith [1, p. 141 § 348] introduced this result *ad hoc* without convincing justification. He used this result as starting point for his deduction of Newton's statement. The proportion (53) is Smith's Prop. III Corol. 1 [1, § 349 p. 141], i.e. in refracting and reflecting telescopes a given object appears equally bright when their aperture diameters are as their linear magnifications.

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